PROBLEMS AND SOLUTIONS 2

Problem 5.2:1

Statement. Find the solution of

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad x, t \in \mathbb{R}, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \end{cases}$$

in the following cases:

(b) $f(x) = e^{-x^2}$, $g(x) = 2axe^{-x^2}$,
(d) $f(x) = 1$, $g(x) = 0$,
(f) $f(x) = 0$, $g(x) = \sin^2 x$.

Solution. We can directly apply D'Alambert's formula

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\mathrm{d}s.$$

(b) We have

$$\int 2xe^{-x^2}\mathrm{d}x = -e^{-x^2} + C,$$

hence

$$\begin{aligned} u(x,t) &= \frac{e^{-(x+at)^2} + e^{-(x-at)^2}}{2} + \frac{1}{2a}\int_{x-at}^{x+at} 2ase^{-s^2}\mathrm{d}s \\ &= \frac{e^{-(x+at)^2} + e^{-(x-at)^2}}{2} + \frac{1}{2}(-e^{-(x+at)^2} + e^{-(x-at)^2}) \\ &= e^{-(x-at)^2}. \end{aligned}$$

(d) $u(x,t) = 1$.
(f) We have

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \sin^2 s\,\mathrm{d}s = \frac{1}{4a}\int_{x-at}^{x+at}(1-\cos 2s)\mathrm{d}s = \frac{1}{2}t + \frac{\sin 2(x-at) - \sin 2(x+at)}{8a}.$$

Problem 5.2:4

Statement. Solve

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 \le x < \infty, \quad -\infty < t < \infty, \\ u_x(0,t) = 0, \\ u(x,0) = x^3, \quad u_t(x,0) = 0. \end{cases}$$

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Solution. We need first to extend the initial data by even reflection to the entire real line $-\infty < c < \infty$, apply D'Alambert's formula to solve the problem on the line, and finally restrict to the half line $x \geq 0$. The even reflection of the initial datum x^3 is $|x|^3$. Let us apply the D'Alambert formula

$$u(x,t) = \frac{|x-at|^3 + |x+at|^3}{2}.$$

Note that since $f(x) = |x|^3$ is a C^2 function, $u(x,t)$ satisfies the wave equation for all x and t. We can calculate

$$u_x(x,t) = \frac{3(x-at)|x-at| + 3(x+at)|x+at|}{2},$$

which implies that $u_x(0,t) = 0$ for all t. To solve the original problem, we just need to restrict our attention to the region $x \geq 0$.

Problem 5.1:1

Statement. Solve the problem

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 \leq x \leq L, \, -\infty < t < \infty, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \end{cases}$$

in the following cases

(a) $f(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L})$, $g(x) = \frac{1}{2}\sin(\frac{2\pi x}{L})$,
(b) $f(x) = \sin^3(\frac{\pi x}{L})$, $g(x) = 0$,
(c) $f(x) = 0$, $g(x) = \sin(\frac{\pi x}{L})\cos^2(\frac{\pi x}{L})$,
(d) $f(x) = \sin^3(\frac{\pi x}{L})$, $g(x) = \sin(\frac{\pi x}{L})\cos^2(\frac{\pi x}{L})$.

Solution. If the initial data satisfy

$$f(x) = \sum_{n=1}^{N} \alpha_n \sin(\frac{n\pi x}{L}), \qquad g(x) = \sum_{n=1}^{N} \beta_n \sin(\frac{n\pi x}{L}),$$

then the solution is given by

$$u(x,t) = \sum_{n=1}^{N} \left(\alpha_n \cos(\frac{n\pi a t}{L}) + \beta_n \frac{L}{n\pi a} \sin(\frac{n\pi a t}{L}) \right) \sin(\frac{n\pi x}{L}).$$

Applying this formula, we get the following.

(a) $u(x,t) = 3\cos(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) - \cos(\frac{4\pi a t}{L})\sin(\frac{4\pi x}{L}) + \frac{L}{4\pi a}\sin(\frac{2\pi a t}{L})\sin(\frac{2\pi x}{L})$.
(b) From the triple angle formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we have $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$, hence $u(x,t) = \frac{3}{4}\cos(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) - \frac{1}{4}\cos(\frac{3\pi a t}{L})\sin(\frac{3\pi x}{L})$.
(c) Again using the triple angle formula $g(x) = \sin(\frac{\pi x}{L}) - \sin^3(\frac{\pi x}{L}) = \frac{1}{4}\sin(\frac{\pi x}{L}) + \frac{1}{4}\sin(\frac{3\pi x}{L})$, which leads to $u(x,t) = \frac{L}{4\pi a}\sin(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) + \frac{L}{12\pi a}\sin(\frac{3\pi a t}{L})\sin(\frac{3\pi x}{L})$.
(d) By combining (b) and (c) above, we infer

$$u(x,t) = \left(\tfrac{3}{4}\cos(\tfrac{\pi a t}{L}) + \tfrac{L}{4\pi a}\sin(\tfrac{\pi a t}{L})\right)\sin(\tfrac{\pi x}{L}) + \left(-\tfrac{1}{4}\cos(\tfrac{3\pi a t}{L}) + \tfrac{L}{12\pi a}\sin(\tfrac{3\pi a t}{L})\right)\sin(\tfrac{3\pi x}{L}).$$

PROBLEM 5.1:6

Statement. Consider the problem

$$\begin{cases} u_{tt} = u_{xx}, \quad 0 \le x \le \pi, \quad -\infty < t < \infty, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \\ u(x,0) = x(\pi - x), \quad u_t(x,0) = 0. \end{cases}$$

(a) Find a function that satisfies the equation and the boundary conditions exactly, and the initial condition to within an error of .001.

(b) By computing u_{tt} and u_{xx} at $(x,t) = (0,0)$, show that there is no C^2 solution of the problem.

Solution. (a) Let us first compute the sine series coefficients for $f(x) = x(\pi - x)$, which are given by

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx \,\mathrm{d}x, \qquad n = 1, 2, \ldots.$$

We need the integrals

$$\int_0^\pi x \sin nx \,\mathrm{d}x = -\left.\frac{x\cos nx}{n}\right|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx \,\mathrm{d}x = \frac{(-1)^{n+1}\pi}{n},$$

and

$$\begin{aligned} \int_0^\pi x^2 \sin nx \,\mathrm{d}x &= -\left.\frac{x^2\cos nx}{n}\right|_0^\pi + \frac{2}{n}\int_0^\pi x\cos nx \,\mathrm{d}x \\ &= \frac{(-1)^{n+1}\pi^2}{n} + \left.\frac{2}{n^2}x\sin nx\right|_0^\pi - \frac{2}{n^2}\int_0^\pi \sin nx \,\mathrm{d}x \\ &= \frac{(-1)^{n+1}\pi^2}{n} + \left.\frac{2\cos nx}{n^2}\right|_0^\pi = \frac{(-1)^{n+1}\pi^2}{n} + \frac{2((-1)^n - 1)}{n^2}. \end{aligned}$$

Hence

$$b_n = (-1)^{n+1}\frac{2\pi}{n} - (-1)^{n+1}\frac{2\pi}{n} + \frac{4(1-(-1)^n)}{n^2\pi} = \frac{4(1-(-1)^n)}{n^2\pi},$$

and the sine series for f is

$$f(x) = \frac{8}{\pi}\sum_{m=0}^\infty \frac{\sin(2m+1)x}{(2m+1)^2}.$$

Obviously, for any $M > 0$, the function

$$u_M(x,t) = \frac{8}{\pi}\sum_{m=0}^M \frac{\sin(2m+1)x}{(2m+1)^2}\cos(2m+1)t,$$

satisfies the wave equation $u_{tt} = u_{xx}$ and the boundary conditions $u(0,t) = u(\pi,t) = 0$. We just need to choose M so large that $|u_M(x,0) - f(x)| \le .001$ for all $0 \le x \le \pi$. We can estimate this error as follows:

$$\begin{aligned} |f(x) - u_M(x,0)| &\le \frac{8}{\pi}\left|\sum_{m=M+1}^\infty \frac{\sin(2m+1)x}{(2m+1)^2}\right| \le \frac{8}{\pi}\sum_{m=M+1}^\infty \frac{1}{(2m+1)^2} \\ &\le \frac{4}{\pi}\int_{2M+1}^\infty \frac{\mathrm{d}\theta}{\theta^2} = \frac{4}{(2M+1)\pi}. \end{aligned}$$

Therefore, to ensure $|u_M(x,0) - f(x)| \le .001$, it is sufficient to take $M \ge \frac{2000}{\pi} - \frac{1}{2}$, i.e., $M \ge 637$.

Problem 5.1:9

Statement. Use separation of variables to find *all* product solutions of the problem

$$\begin{cases} u_{tt} = a^2 u_{xx} - k u_t, \quad 0 \le x \le L, \quad -\infty < t < \infty, \\ u(0,t) = 0, \quad u(L,t) = 0, \end{cases}$$

for the string with air resistance and fixed ends (assume $k > 0$).

Solution. Putting $u(x,t) = X(x)T(t)$, we have

$$XT'' = a^2 X'' T - kXT',$$

and division by XT gives

$$\frac{T''}{T} + k\frac{T'}{T} = a^2 \frac{X''}{X} = a^2 \alpha,$$

where α is a constant. If $\alpha = 0$, then the equation for X becomes $X'' = 0$, meaning that $X(x) = Ax + B$. But the boundary conditions $X(0) = X(L) = 0$ forces $X(x) \equiv 0$. So this case is trivial. Now if $\alpha = \lambda^2 > 0$ with $\lambda > 0$, we have

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}.$$

The boundary conditions give $A + B = 0$ and $Ae^{\lambda} + Be^{-\lambda} = 0$, implying that $A = B = 0$. Finally, consider the remaining case $\alpha = -\lambda^2 < 0$ with $\lambda > 0$. The general solution for X is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x),$$

and from $X(0) = 0$ we immediately get $A = 0$. Then $X(L) = B\sin(\lambda L) = 0$ gives the condition $\lambda = \frac{n\pi}{L}$ for some positive integer n. To conclude the analysis of the equation for X, the only solutions are

$$X_n(x) = \sin(\frac{n\pi x}{L}), \qquad n = 1, 2, \ldots,$$

and their linear combinations.

We shall consider the equation for T. With $\alpha = -\lambda^2 = -\frac{n^2\pi^2}{L^2}$, we have

$$T'' + kT' + \omega_n^2 T = 0,$$

where $\omega_n = \frac{n\pi a}{L}$. This (standard equation for damped oscillator) can easily be solved by the ansatz $T(t) = e^{\mu t}$, which yields

$$\mu = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \omega_n^2}.$$

If $\omega_n < \frac{k}{2}$, we have two monotone solutions

$$T_n(t) = Ae^{-k_n^+ t} + Be^{-k_n^- t}, \qquad k_n^{\pm} = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \omega_n^2}.$$

If $\omega_n > \frac{k}{2}$, we have the oscillating solutions

$$T_n(t) = e^{-kt/2}(A\cos\tilde{\omega}_n t + B\sin\tilde{\omega}_n t), \qquad \tilde{\omega}_n = \sqrt{\omega_n^2 - \frac{k^2}{4}}.$$

If it so happens that $\omega_n = \frac{k}{2}$, we have

$$T_n(t) = e^{-kt/2}(A + Bt).$$

To conclude, all product solutions of the given problem are given by

$$u(x,t) = T_n(x)\sin(\frac{n\pi x}{L}),$$

as n ranges over the positive integers, where T_n is one of the above three functions depending on how $\omega_n = \frac{n\pi a}{L}$ compares with $\frac{k}{2}$. Note that given n, T_n is one and only one of the above three choices.