PROBLEMS AND SOLUTIONS 1

Problems 1.3:2d and 1.3:3d

Statement. Find general solutions of $yu_{xy} + 2u_x = x$ using ODE techniques, as well as its particular solution satisfying the side conditions $u(x, 1) = 0$ and $u(0, y) = 0$.

Solution. Integrating the equation with respect to x gives

$$yu_y + 2u = \frac{x^2}{2} + C(y),$$

where $C(y)$ is an arbitrary function of y. Now we use the integrating factor $m(y) = y$ to proceed

$$(y^2u)_y = y^2u_y + 2yu = y(yu_y + 2u) = \frac{1}{2}x^2y + yC(y).$$

Hence we have

$$y^2u(x, y) = \frac{1}{4}x^2y^2 + \int yC(y)\mathrm{d}y + B(x),$$

and the solution is

$$u(x, y) = \frac{1}{4}x^2 + A(y) + y^{-2}B(x),$$

where A and B are differentiable functions. This is indeed a solution, since it gives

$$u_x(x, y) = \frac{1}{2}x + y^{-2}B'(x), \qquad \text{and} \qquad u_{xy}(x, y) = -2y^{-3}B'(x).$$

Now let us plug in the side conditions. We get

$$u(x, 1) = \frac{1}{4}x^2 + A(1) + B(x) = 0, \qquad \Rightarrow \qquad B(x) = -A(1) - \frac{1}{4}x^2,$$

and

$$u(0, y) = A(y) + y^{-2}B(0) = 0, \qquad \Rightarrow \qquad A(y) = -y^{-2}B(0) = y^{-2}A(1),$$

leading to the final solution

$$u(x, y) = \frac{1}{4}x^2 + y^{-2}A(1) + y^{-2}\left(-A(1) - \frac{1}{4}x^2\right) = \frac{x^2}{4}\left(1 - \frac{1}{y^2}\right).$$

Problem 2.1:2b

Statement. Find the particular solution of $u_x + 2u_y - 4u = e^{x+y}$ satisfying $u(0, y) = y^2$.

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Solution. To find the general solution, let us use the coordinate transformation

$$x = \xi, \qquad y = \eta + 2\xi.$$

The variable y transforms into the new variable η, and x stays the same. The useful property of this transformation is

$$u_\xi = u_x x_\xi + u_y y_\xi = u_x + 2u_y,$$

which converts the equation into

$$u_\xi - 4u = e^{3\xi+\eta}.$$

This can be solved by using the integrating factor $m(\xi) = e^{-4\xi}$ as

$$(e^{-4\xi}u)_\xi = e^{-4\xi}(u_\xi - 4u) = e^{\eta-\xi},$$

leading to

$$e^{-4\xi}u(\xi,\eta) = -e^{\eta-\xi} + C(\eta), \qquad \Rightarrow \qquad u(\xi,\eta) = -e^{3\xi+\eta} + e^{4\xi}C(\eta).$$

Taking into account that $\xi = x$ and $\eta = y - 2x$, we have

$$u(x,y) = -e^{x+y} + e^{4x}C(y-2x).$$

Now plugging the side condition into this, we get

$$u(0,y) = -e^y + C(y) = y^2, \qquad \Rightarrow \qquad C(y) = y^2 + e^y.$$

So the final solution is

$$u(x,y) = -e^{x+y} + e^{4x}\left((y-2x)^2 + e^{y-2x}\right).$$

Problem 2.1:3

Statement. Show that the PDE $u_x + u_y - u = 0$ with the side condition $u(x,x) = \tan(x)$ has no solution.

Solution. An intuitive explanation of the situation is that the side condition is given on a characteristic line, and is inconsistent with the differential equation. Let us define the function $f(t) = u(x(t), y(t))$, where $x(t) = t$ and $y(t) = t$. We calculate the derivative of this function from the PDE as

$$\begin{aligned} f'(t) &= u_x(x(t),y(t))x'(t) + u_y(x(t),y(t))y'(t) \\ &= u_x(x(t),y(t)) + u_y(x(t),y(t)) \\ &= u(x(t),y(t)) \\ &= f(t), \end{aligned}$$

which means that $f(t) = f(0)e^t$. From the side condition, we have $f(t) = \tan(t)$, which clearly does not satisfy $f' = f$. Hence, there is no function f that satisfies the both conditions. As f is simply the value of $u(x,y)$ on the "diagonal" $\{x = y\}$, this makes the existence of u impossible.

Problem 2.1:8

Statement. (a) Show that the PDE $u_x = 0$ has no solution which is C^1 everywhere and satisfies the side condition $u(x,x^2) = x$.

(b) Find a solution of the problem in (a) which is valid in the first quadrant $x > 0$, $y > 0$.

(c) Explain the results of (a) and (b) in terms of the intersections of the side condition curve and the characteristic lines.

Solution. (a) The PDE simply means that u does not depend on x, hence the general solution is $u(x,y) = g(y)$ for any function g. Similarly to the previous problem, let us define the function $f(t) = u(x(t), y(t))$ with $x(t) = t$ and $y(t) = t^2$. From the PDE, we have $f(t) = g(t^2) = g((-t)^2) = f(-t)$, meaning that f is an even function. However, the side condition gives $f(t) = t$, which is a nontrivial odd function. Hence, there is no function f that satisfies the both conditions.

(b) Enforcing the side condition to the general solution $u(x,y) = g(y)$ gives $u(x,x^2) = g(x^2) = x$, and since $x > 0$ we get $g(y) = \sqrt{y}$ for $y > 0$. So the solution is $u(x,y) = \sqrt{y}$ for $x > 0$ and $y > 0$.

(c) The characteristic lines are simply the horizontal lines $\{y = \text{const}\}$, and the PDE requires the solutions to be constant along those lines. The side condition curve is the parabola $y = x^2$, and it intersects each characteristic line in the upper half plane $\{y > 0\}$ twice. This means that in order for a solution to exist, the side condition must be symmetric with respect to the y-axis. However, the side condition $u(x,x^2) = x$ does not have the required symmetry.

Problem 2.1:9

Statement. (a) Show that the PDE $u_x = 0$ has no solution which is C^1 everywhere and satisfies the side condition $u(x,x^3) = x$, even though the side condition curve $y = x^3$ intersects each characteristic lines.

(b) Part (a) demonstrates the necessity of the transversality condition on the intersections of the side condition curve with the characteristic lines. Explain why.

Solution. (a) Let us define the function $f(t) = u(x(t), y(t))$ with $x(t) = t$ and $y(t) = t^3$. From the PDE, the derivative of this function is

$$\begin{aligned} f'(t) &= u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) \\ &= u_x(x(t), y(t)) + 3t^2 u_y(x(t), y(t)) \\ &= 3t^2 u_y(x(t), y(t)), \end{aligned}$$

so in particular, $f'(0) = 0$. The side condition gives $f(t) = t$, implying that $f'(t) = 1$ for all t. Hence, there is no function f that satisfies the both conditions. Note that the same reasoning would have worked also for the previous problem.

(b) Consider a situation where a characteristic curve intersects the side condition curve twice, and imagine that the two intersection points are getting closer and closer together, as a result of varying the side condition curve in some way. In the limit it would produce a situation where a characteristic curve intersects the side condition curve tangentially. Since in the case with two intersections the solution u must have the same value at the two points, intuitively speaking, in the limit case the tangential derivative of u at the intersection point must vanish. So if the tangential derivative of the side condition at the intersection point is nonzero, a contradiction would arise.

Problem 2.1:10

Statement. (a) Show that a solution of the homogeneous PDE $au_x + bu_y + cu = 0$ cannot be zero at one, and only one, point on the plane.

(b) If $c = 0$ in the PDE in (a), then show that the graph $z = u(x,y)$ of a solution u (defined everywhere) is a surface composed of horizontal parallel lines.

Solution. We will assume that at least one of a and b is nonzero, i.e., $a^2+b^2 > 0$. If $a = b = 0$, then the equation reduces to $cu = 0$, which implies $u \equiv 0$ provided $c \neq 0$. There is no point in considering the case $a = b = c = 0$.

(a) Suppose that u is a solution of the PDE, and suppose that $u(x_0, y_0) = 0$ at some point (x_0, y_0). We are done if we can produce a point (x, y) such that $u(x, y) = 0$ and $(x, y) \neq (x_0, y_0)$. Let us define the function $f(t) = u(x(t), y(t))$, with $x(t) = x_0 + at$ and $y(t) = y_0 + bt$. It is obvious that the point $(x(t), y(t))$ is different from (x_0, y_0) unless $t = 0$, and coincides with (x_0, y_0) if $t = 0$. In particular, we have $f(0) = 0$. Let us calculate the derivative of f, by using the PDE as

$$\begin{aligned} f'(t) &= u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) \\ &= au_x(x(t), y(t)) + bu_y(x(t), y(t)) \\ &= -cu(x(t), y(t)) \\ &= -cf(t). \end{aligned}$$

This implies that $f(t) = f(0)e^{-ct}$, but as $f(0) = 0$, we get $f(t) = 0$ for all t. Thus for example, choosing $t = 1$ gives $f(1) = u(x(1), y(1)) = u(x_0 + a, y_0 + b) = 0$.

(b) Let (x_0, y_0) be an arbitrary point on the plane, and define the function f as in (a). Then we have $f'(t) = -cf(t)$, and since $c = 0$, we get $f(t) = \text{const}$ for all t. Note that the derivation of the equation $f' = -cf$ in (a) does not depend on the assumption $u(x_0, y_0) = 0$, and the latter assumption was only used to infer $f(0) = 0$. Now $f(t) = \text{const}$ means in terms of u that $u(x_0 + at, y_0 + bt) = u(x_0, y_0)$ for all t. The graph Γ of u is the surface in $\mathbb{R}^3$ defined by

$$\Gamma = \{(x, y, u(x, y)) : (x, y) \in \mathbb{R}^2\}.$$

The above consideration says that if the point (x_0, y_0, z_0) is in the graph, that is to say if $z_0 = u(x_0, y_0)$, then the line $\ell(x_0, y_0) = \{(x_0 + at, y_0 + bt, z_0) : t \in \mathbb{R}\}$ is entirely contained in the graph. We see that the graph Γ is simply the union of all the lines $\ell(x_0, y_0)$ as the point (x_0, y_0) runs over the plane $\mathbb{R}^2$.

Problems 2.2:1d and 2.2:2d

Statement. Obtain the general solution of $yu_x - 4xu_y = 2xy$ for all (x, y), as well as its particular solution satisfying the side condition $u(x, 0) = x^4$.

Solution. The characteristic equation is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{y}.$$

As this equation is separable, we can integrate it to get

$$y^2 + 4x^2 = r,$$

which describes an ellipse for each value of the constant $r \geq 0$. Differentiation of the solution u along one of those characteristic curves gives

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x, y(x)) = u_x + u_y y_x = u_x - \frac{4x}{y}u_y = \frac{yu_x - 4xu_y}{y} = 2x,$$

and upon integration we have

$$u(x, y(x)) = x^2 + f(r),$$

where $f(r)$ is an arbitrary C^1 function of $r \geq 0$. Note here that the function $y(x)$ depends implicitly on the parameter r, which parameterizes the space of all characteristic curves. Now, given an arbitrary point (x, y) on the plane $\mathbb{R}^2$, clearly it lies on the characteristic curve with the parameter $r = y^2 + 4x^2$. Hence we can write the general solution as

$$u(x, y) = x^2 + f(y^2 + 4x^2). \tag{1}$$

The calculation

$$u_x(x,y) = 2x + 8xf'(y^2+4x^2), \qquad u_y(x,y) = 2yf'(y^2+4x^2), \tag{2}$$

makes it clear that it is indeed a solution of the PDE $yu_x - 4xu_y = 2xy$.

To find the particular solution, we enforce the side condition

$$u(x,0) = x^2 + f(4x^2) = x^4,$$

which, under the substitution $z = 4x^2$, yields

$$f(z) = \frac{z^2}{16} - \frac{z}{4} = \frac{z(z-4)}{16}, \qquad (z \geq 0).$$

Substituting this back into (1), we conclude

$$u(x,y) = x^2 + \frac{(y^2+4x^2)^2}{16} - \frac{y^2+4x^2}{4} = \frac{(y^2+4x^2)^2 - 4y^2}{16}.$$

Problem 2.2:3d

Statement. Find the parametric form of the solution of $yu_x - 4xu_y = 2xy$, which satisfy the side condition $u(s,s^3) = 1$.

Solution. The characteristic equation in parametric form is

$$x_t = y, \qquad y_t = -4x. \tag{3}$$

We differentiate the first equation, and substitute y_t from the second, to get

$$x_{tt} = y_t = -4x.$$

The general solution of this ODE is

$$x(t) = A\cos(2t) + B\sin(2t), \qquad \Rightarrow \qquad y(t) = x_t(t) = -2A\sin(2t) + 2B\cos(2t).$$

For every pair of constants A and B, the pair of functions $x(t)$ and $y(t)$ gives one characteristic curve (as t runs over the real numbers). We want to choose A and B so that the parameter value $t = 0$ corresponds to an intersection point between the characteristic curve and the side condition curve. Since different characteristic curves intersect the side condition curve at different points, this offers us a possibility to give identities to the characteristic curves by where they intersect the side condition curve. In turn, this means that the parameterization (with the parameter s) of the side conditions curve can be used to label the characteristic curves. Now we proceed to implement these ideas. We use the functions $x(s,t)$ and $y(s,t)$ to describe the family of characteristic curves: for a fixed value of s we have a characteristic curve as t runs over the real numbers, and different values of s would correspond to different characteristic curves. Thus we have

$$x(s,t) = A_s\cos(2t) + B_s\sin(2t), \qquad y(s,t) = -2A_s\sin(2t) + 2B_s\cos(2t),$$

with the constants A_s and B_s depending on s. Their dependence on s should be inferred from the conditions

$$x(s,0) = s, \qquad y(s,0) = s^3, \tag{4}$$

which gives $A_s = s$ and $B_s = s^3/2$, and so

$$x(s,t) = s\cos(2t) + \tfrac{1}{2}s^3\sin(2t), \qquad y(s,t) = -2s\sin(2t) + s^3\cos(2t). \tag{5}$$

Let us calculate the t-derivative of the solution u along the characteristic curves

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u(x(s,t),y(s,t)) &= y(s,t)u_x(x(s,t),y(s,t)) - 4x(s,t)u_y(x(s,t),y(s,t))\\ &= 2x(s,t)y(s,t)\\ &= \frac{s^6-4s^2}{4}\sin(4t) + s^4\cos(4t),\end{aligned}$$

where we have taken into account (3) in the first line, used the PDE $yu_x - 4xu_y = 2xy$ in the second line, and finally substituted (5) in the third line. We can integrate this in t and get

$$U(s,t) = \frac{s^4}{4}\sin(4t) + \frac{4s^2-s^6}{16}\cos(4t) + C(s), \tag{6}$$

where we have used the notation $U(s,t) = u(x(s,t),y(s,t))$.

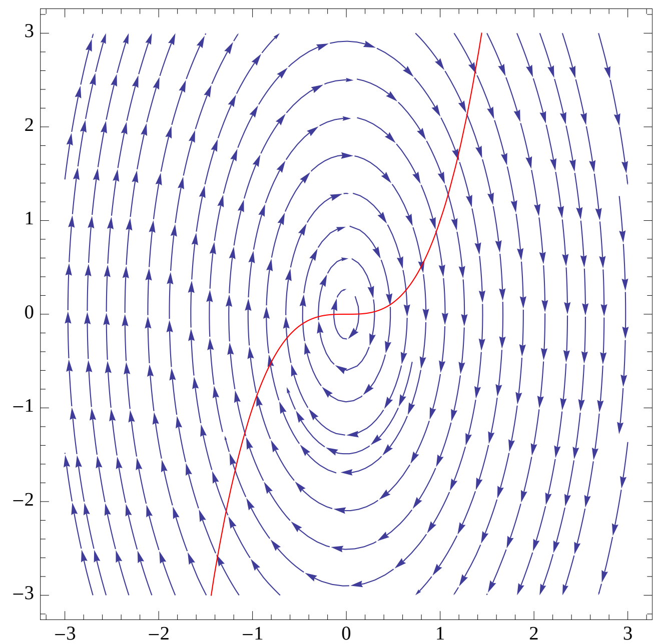

FIGURE 1. Characteristic curves and the side condition curve for Problem 2.2:3(d).

Now let us step back and examine the geometry of the parameterization we have obtained. We know that as t runs over the real numbers with s fixed, the point $P(s,t) = (x(s,t),y(s,t))$ traces out an ellipse centred at the origin. The functions $\sin(2t)$ and $\cos(2t)$ both have period π, so for example, t running from t_0 to $t_0+\pi$ corresponds to the point $P(s,t)$ making one full ellipse around the origin, in the clockwise direction. Each characteristic curve (except the trivial one consisting of the origin only) intersects the side condition curve $y = x^3$ exactly twice. Put another way, for $s \neq 0$ and t real number, the equation $P(-s,t) = P(s,t+\tau)$ is satisfied if and only if $\tau = \pi n + \frac{1}{2}\pi$ for some integer n. This means that the function $U(s,t)$ from (6) defines a function of x and y if and only if it satisfies $U(-s,t) = U(s,t+\pi n+\frac{1}{2}\pi)$ for any integer n. It is a quite restrictive condition, but fortunately the first two terms in (6) are even in s and periodic in t with period $\frac{1}{2}\pi$, reducing the condition to $C(-s) = C(s)$. To conclude, there exists a solution of $yu_x - 4xu_y = 2xy$ satisfying the side condition $U(s,0) = \gamma(s)$ if and only if $\gamma(s)$ gives rise to C in (6) that is even.

It remains to check whether or not our side condition $U(s,0) = 1$ leads to an even C. Enforcing the side condition gives $C(s) = 1 - (4s^2 - s^6)/16$, or

$$U(s,t) = \frac{s^4}{4}\sin(4t) + \frac{4s^2 - s^6}{16}(\cos(4t) - 1) + 1.$$

Since C is an even function, the above expression defines a solution of the problem.

Problem 2.2:4

Statement. Show that the PDE $yu_x - 4xu_y = 2xy$ has no solution satisfying the side condition $u(x,0) = x^3$. Explain the result in terms of characteristic curves.

Solution. The equation is the same as in the preceding problem, except that the side condition is the function x^3 on the x-axis. We proceed as in the preceding solution to find a parameterization of the characteristic curves, keeping in mind that (4) should be replaced by

$$x(s,0) = s, \qquad y(s,0) = 0,$$

which gives $A_s = s$ and $B_s = 0$, and so

$$x(s,t) = s\cos(2t), \qquad y(s,t) = -2s\sin(2t).$$

The equation for the t-derivative of u is

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(s,t), y(s,t)) = 2x(s,t)y(s,t) = -2s^2\sin(4t),$$

implying that

$$U(s,t) = \frac{s^2}{2}\cos(4t) + C(s),$$

where $U(s,t) = u(x(s,t), y(s,t))$. The same argument as in the preceding solution gives the compatibility condition that C must be even. However, the side condition $U(s,0) = s^3$ leads to $C(s) = s^3 - s^2/2$, which is not even, hence the nonexistence.

Each nontrivial characteristic curve, which is an ellipse centred at the origin, intersects the x-axis twice. So not all side conditions would give rise to a solution. In order for a solution to exist, the side condition must be compatible with the behaviour of the solution along the characteristic curves dictated by the differential equation. In this particular case, we see that the side condition is not compatible with the PDE.

Problem 2.2:5

Statement. Show that the only solutions of the PDE $xu_x + 2yu_y = 0$ that are C^1 and defined for all (x,y) are the constant functions (e.g., $u(x,y) = 5$). *Hint*: Observe that the characteristic curves all issue from the origin.

Solution. Suppose that u is a C^1 function which satisfies our equation on the whole plane $\mathbb{R}^2$, and let $A = u(0,0)$. Looking for the characteristic curves in parametric form, we discover that they are

$$x(t) = x_0e^t, \qquad y(t) = y_0e^{2t}, \tag{7}$$

where x_0 and y_0 are constants. Pick an arbitrary point (x_0, y_0) on the plane $\mathbb{R}^2$, and consider the characteristic curve (7) going through it. The PDE tells us that u must be constant along this curve. Let $B = u(x_0, y_0)$ be the value of u on this curve. By definition of continuity, for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x^2 + y^2 < \delta^2$ we have $|u(x,y) - A| < \varepsilon$. In words, by choosing the point (x,y) close enough to the origin, we can make $u(x,y)$ arbitrarily close to the value $u(0,0)$. Now let us observe that $(x(t), y(t))$ in (7) tends to the origin as $t \to -\infty$, implying that for any $\delta > 0$ there is $t_\delta \in \mathbb{R}$ such that $x(t_\delta)^2 + y(t_\delta)^2 < \delta^2$. In other words, the characteristic curve (7) contains points that are arbitrarily close to the origin.

Combined with the above discussion about continuity, this means that for any $\varepsilon > 0$ we have $|B - A| < \varepsilon$. Hence $u(x_0, y_0) = B = A$, and as (x_0, y_0) is an arbitrary point, we conclude that u is constant thoughout the plane.

Problem 2.2:8

Statement. Consider the PDE $\sin(x)u_x - y\cos(x)u_y = 0$.

(a) Sketch the characteristic curves of this PDE.

(b) Show that any regular side condition curve, which transversely (i.e., at a nonzero angle) intersects, exactly once, any characteristic curve of this PDE that it encounters, must be contained in a vertical strip of width 2π.

(c) Deduce that infinitely many side condition curves are needed in order to uniquely determine a solution of this PDE, which is defined throughout the xy-plane.

(d) Show that, given an infinite family of C^1 functions, say $f_n(y)$, such that $f_n(0) = 0$ and $f_n'(0) = 0$ $(n = 0, \pm 1, \pm 2, \ldots)$, there is a solution $u(x, y)$ (C^1 for all (x, y)) of the PDE which satisfies each of the infinitely many side conditions

$$u(n\pi + \tfrac{1}{2}\pi, y) = f_n(y), \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots.$$

Solution. (a) The characteristic equation is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -y\cot x.$$

which can be integrated (somewhat sloppily) as

$$\log y = -\log\sin x + C, \qquad \Rightarrow \qquad y(x) = \frac{A}{\sin x},$$

for some constant A. To make up for the sloppiness, we can confirm

$$y'(x) = -\frac{A\cos x}{\sin^2 x} = -y(x)\cot x,$$

provided $\sin x \neq 0$, i.e., $x \neq n\pi$ for any integer n. If $x = n\pi$ for some integer n, we can resort to the characteristic equation in parametric form

$$x_t = 0, \qquad y_t = -y,$$

and see that the vertical half lines $\{(n\pi, y) : y > 0\}$ and $\{(n\pi, y) : y < 0\}$, as well as the points $(n\pi, 0)$, where n is an integer, are characteristic curves. To summarize, for each integer n, we have the following set of characteristic curves:

i) The curve $y(x) = A/\sin x$, where $n\pi < x < n\pi + \pi$, for each real number A;
ii) The vertical half lines $\{(n\pi, y) : y > 0\}$ and $\{(n\pi, y) : y < 0\}$;
iii) The point $(n\pi, 0)$.

The characteristic curves are sketched in Figure 2.

(b) Assume the contrary, i.e., assume that there is a regular curve Γ, which transversely intersects, exactly once, any characteristic curve that it encounters, which is not contained in a vertical strip of width 2π. Let $(\alpha(s), \beta(s))$ be a parameterization of Γ. Our assumption means that there are parameter values s_0 and s_1 such that $|\alpha(s_0) - \alpha(s_1)| \geq 2\pi$. Without loss of generality we can assume $\alpha(s_0) < \alpha(s_1)$ (otherwise interchange the roles of s_0 and s_1). Then the interval $[\alpha(s_0), \alpha(s_1)]$, being an interval of length greater than 2π, must contain an interval of the form $[n\pi, n\pi + \pi]$ for some integer n. Since the function $\alpha(s)$ is continuous, there are parameter values s_3 and s_4 such that $\alpha(s_3) = n\pi$ and $\alpha(s_4) = n\pi + \pi$. Without loss of generality, we can assume that $s_3 < s_4$ and that $\alpha(s) \in [n\pi, n\pi + \pi]$ for all $s \in [s_3, s_4]$.

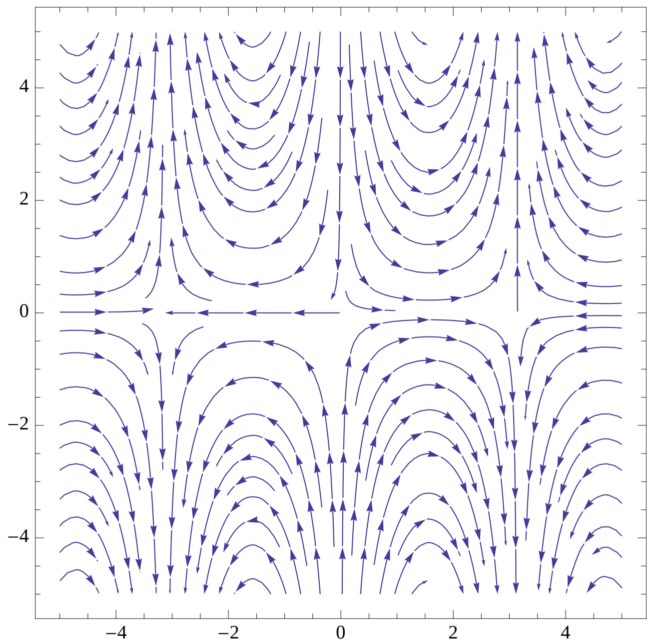

FIGURE 2. Characteristic curves for Problem 2.2:8.

Let us consider first the case $\beta(s_3) > 0$. Since the half line $\{(n\pi, y) : y > 0\}$ is a characteristic curve, the transversality condition gives $\alpha'(s_3) \neq 0$. This means that Γ is described by the equation

$$y(x) = \beta(s_3) + \frac{\beta'(s_3)}{\alpha'(s_3)}(x - n\pi) + g(x - n\pi),$$

for x close to $n\pi$, with some C^1 function g satisfying $g(0) = g'(0) = 0$. Given a point (x_0, y_0) on Γ, say, with $y_0 > 0$ and $x_0 > n\pi$ but $x_0 - n\pi$ small, the characteristic curve

$$y(x) = \frac{A}{\sin x}, \qquad \text{with} \qquad A = y_0 \sin x_0, \tag{8}$$

goes through the point (x_0, y_0). At that point, the slope of $y(x) = A/\sin x$ is $y'(x_0) = -y_0 \cot x_0$, which goes to $-\infty$ as (x_0, y_0) approaches $(n\pi, \beta(s_3))$. Hence there exist points (x_0, y_0) and (x_*, y_*) on Γ with $x_* > x_0 > n\pi$ and $x_* - n\pi$ small, such that

$$y_* > \frac{y_0 \sin x_0}{\sin x_*},$$

that is, the characteristic curve that intersects Γ at (x_0, y_0) goes *below* Γ for the values of x that are immediately right to x_0. Let us denote by s_0 and s_* the parameter values of Γ corresponding to the points (x_0, y_0) and (x_*, y_*). Then the function

$$f(s) = \beta(s) - \frac{y_0 \sin x_0}{\sin \alpha(s)},$$

satisfies

$$f(s_0) = 0, \qquad \text{and} \qquad f(s_*) > 0.$$

Now, as $s \to s_4$, we have $\beta(s) \to \beta(s_4)$, but $\frac{y_0 \sin x_0}{\sin \alpha(s)} \to \infty$, since $\alpha(s) \in (n\pi, n\pi + \pi)$ and $\alpha(s) \to n\pi + \pi$. This implies that $f(s) \to -\infty$, hence by continuity there is $s_{**} \in (s_*, s_4)$ such that $f(s_{**}) = 0$, meaning that the characteristic curve (8) intersects Γ (at least) twice (once at s_0, and once at s_{**}). We have established the desired contradiction.

The case $\beta(s_3) < 0$ can be treated in the same way. If $\beta(s_3) = 0$ but $\beta(s_4) \neq 0$, we can still use the same approach, by replacing the roles of s_3 and s_4.

What is left is only the case $\beta(s_3) = \beta(s_4) = 0$. We can assume the existence of an $s_* \in (s_3, s_4)$ such that $\beta(s_*) \neq 0$, since otherwise Γ would have to contain, therefore intersect more than once, the entire characteristic curve $\{(x, 0) : n\pi < x < n\pi + \pi\}$. Suppose further that $\beta(s_*) > 0$. Then there exists a characteristic curve that crosses the line segment joining the points $(\alpha(s_*), \beta(s_*))$ and $(\alpha(s_*), 0)$, not touching the endpoints. This characteristic curve must approach $+\infty$ as x tends to $n\pi$ or $n\pi + \pi$. But the curve Γ stays bounded (in fact it joins the points $(0, n\pi)$ and $(0, n\pi + \pi)$), which exhibits at least two common points for Γ and the characteristic curve we are considering. The case $\beta(s_*) < 0$ is completely analogous.

(c) Since any reasonable side condition curve must be confined in a vertical strip, there will be characteristic curves that do not intersect the side condition curve at all, meaning that one side condition curve cannot intersect all characteristic curves. One side condition curve would be enough to determine the solution on a vertical strip, but note that we want to determine the solution throughout the xy-plane. Even finitely many side condition curves are not enough, since they would occupy a vertical strip of only a finite width. Therefore we need infinitely many side condition curves.

(d) The side condition curves are the vertical lines $\{n\pi + \frac{1}{2}\pi\} \times \mathbb{R}$ where n is an arbitrary integer. Every characteristic curve of type *i)* from Part (a) intersects exactly one side condition curve, exactly once and transversely. Hence the solution is determined on the vertical strips $(n\pi, n\pi + \pi) \times \mathbb{R}$. We only need to check what happens at the vertical lines $\{n\pi\} \times \mathbb{R}$. Given a point (x_0, y_0) in the strip $(n\pi, n\pi + \pi) \times \mathbb{R}$, the characteristic curve

$$y(x) = \frac{A}{\sin x}, \qquad \text{with} \qquad A = y_0 \sin x_0,$$

goes through the point (x_0, y_0). This characteristic curve intersects the side condition curve $\{(n\pi + \frac{1}{2}\pi, y) : y \in \mathbb{R}\}$ at $y = (-1)^n y_0 \sin x_0$. So at (x_0, y_0), the solution is given by $u(x_0, y_0) = f_n((-1)^n y_0 \sin x_0)$. We can omit the subscripts from x_0 and y_0 and write

$$u(x, y) = f_n((-1)^n y \sin x), \qquad \text{for} \quad n\pi < x < n\pi + \pi.$$

As x approaches $n\pi$ or $n\pi + \pi$, the argument of f_n tends to 0, so (since $f(0) = 0$) by continuity $u(x, y)$ goes to 0. Let us define $u(n\pi, y) = 0$ for all y, which is necessary if u were to be continuous. The partial derivatives of u are given by

$$u_x(x, y) = (-1)^n y \cos x \cdot f_n'((-1)^n y \sin x), \qquad u_y(x, y) = (-1)^n \sin x \cdot f_n'((-1)^n y \sin x),$$

which tend to 0 as x approaches $n\pi$ or $n\pi + \pi$, by continuity of f_n' and the fact that $f_n'(0) = 0$. Note that one does not even need to use the fact $f_n'(0) = 0$ to infer $u_y(x, y) \to 0$. We can now conclude that the solution u is C^1 and is uniquely determined by the side conditions.