MULTIPOLE EXPANSIONS IN THE PLANE

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Contents

1. Electrostatics and gravitation

Newton's *law of universal gravitation*, first published in his *Principia* in 1687, asserts that the force exerted on a point mass Q at $x \in \mathbb{R}^3$ by the system of finitely many point masses q_i at $y_i \in \mathbb{R}^3$, $(i = 1, \ldots, m)$, is equal to

$$F = \sum_{i=1}^{m} \frac{Cq_i Q}{|x - y_i|^2} \frac{x - y_i}{|x - y_i|}, \tag{1}$$

with a constant $C < 0$ (like masses attract). Here Q and q_i are understood as real numbers that measure how much mass the corresponding points have, and $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ is the Euclidean length of the vector $a \in \mathbb{R}^3$. The same law of interaction between point charges was discovered experimentally by Charles Augustin de Coulomb and announced in 1785, now with $C > 0$ (like charges repel). Note that the numerical value of the constant C depends on the unit system one is using to measure force, mass (or charge), and distance.

It is convenient to view the force $F = F(x)$ as a vector function of x, that is, a vector field. This means that we fix the configuration of the point masses $\{q_i\}$, and think of Q as a test mass, that can be placed at any point in space to "probe the field." The vector field

$$E(x) = \sum_{i=1}^{m} \frac{Cq_i}{|x - y_i|^2} \frac{x - y_i}{|x - y_i|}, \tag{2}$$

does not depend on the test mass Q, and given any test mass Q at $x \in \mathbb{R}^3$, the force can be recovered as $F = QE(x)$. Therefore E can be thought of as a preexisting entity that characterizes the gravitational (or electric) field generated by the point masses $\{q_i\}$. In fact, we call E the *gravitational field* (or the *electric field*).

In this short note, we give an introduction to *multipole expansions*, which is a powerful method to get a handle on the field E, when it is generated by a system of large number of particles, or by a body with continuous distribution of mass (or charge). The method was originally developed to calculate the gravitational field of an irregular shaped object, such as the Earth, and is largely due to the three great mathematicians Joseph-Louis Lagrange, Pierre-Simon Laplace, and Adrien-Marie Legendre. It was later extended to treat wave propagation problems in electromagnetism and general relativity.

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For continuous distribution of mass, the sum in (2) must be replaced by an integral, as

$$E(x) = C \int_\Omega \frac{\rho(y)}{|x-y|^2} \frac{x-y}{|x-y|} \mathrm{d}y, \tag{3}$$

where ρ is the mass (or charge) density, and $\Omega \subset \mathbb{R}^3$ is the region occupied by the body. By defining $\rho = 0$ outside Ω, in (3) we may integrate over $\mathbb{R}^3$. Now, the method of multipole expansions depends on a few crucial observations. Firstly, in 1773, Lagrange showed that the field E is (minus) the gradient of some scalar function u, called the *potential*, that is,

$$\text{there is a scalar function } u \text{ such that } E = -\operatorname{grad} u \equiv -\big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\big). \tag{4}$$

Secondly, in 1782, Laplace observed that E is *divergence-free* in empty space, that is,

$$\operatorname{div} E \equiv \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = 0 \qquad \text{in free space}, \tag{5}$$

where *free space* means a place where there is no mass (or charge). From (4) and (5) we see that the potential satisfies

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0 \qquad \text{in free space}, \tag{6}$$

where Δ is called the *Laplace operator*. The equation (6) is the *Laplace equation*, and its solutions are called *harmonic functions*. It should however be noted that the same equation had been considered by Lagrange in 1760 in connection with his study of fluid flow problems.

Laplace's result (6) was completed by his student Siméon Denis Poisson in 1813, when Poisson showed that

$$\Delta u = -4\pi C \rho \qquad \text{in } \mathbb{R}^3, \tag{7}$$

for ρ vanishing outside some bounded set[1]. This equation is called the *Poisson equation*, and is valid everywhere, as opposed to (6), which is only valid in free space. Note that in terms of the field E, the Poisson equation (7) is simply $\operatorname{div} E = 4\pi C\rho$.

Finally, around 1785, Legendre and Laplace devised a method to expand u in a series of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{8}$$

where u_n behaves like $u_n(x) \sim |x|^{-n}$ for $x \in \mathbb{R}^3$ far away from the origin. Then by summing only the first few terms in the right hand side of (8), we can approximate $u(x)$ with high accuracy, especially for $|x|$ large. This expansion was the first instance of what came to be known as multipole expansions.

In the next section, we will confirm the existence of a potential (4), divergence-free property of the field (5), and the Poisson equation (7). Then in the following section, we will derive the multipole expansion (8) in two dimensions.

2. The Laplace equation

In order to confirm the existence of a potential (4), we simply define

$$u(x) = \sum_{i=1}^{m} \frac{Cq_i}{|x-y_i|}, \tag{9}$$

[1] In Gaussian type unit systems, one sets up the units so that $C = \pm 1$, and hence the Newtonian/Coulomb formulas (2) and (3) have simple expressions. In other systems such as SI, one has $C = \pm\frac{1}{4\pi}$, meaning that the Poisson equation (7) has a simple expression.

for a system of point charges, and

$$u(x) = C\int_{\mathbb{R}^3} \frac{\rho(y)\mathrm{d}y}{|x-y|}, \tag{10}$$

for a continuous distribution of charge, and check that $E = -\mathrm{grad}\, u$ indeed gives (2) and (3), respectively. To this end, we compute

$$\frac{\partial}{\partial x_1}\frac{1}{|x|} = \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-1/2} = -\frac{1}{2}\left(|x|^2\right)^{-3/2}\cdot 2x_1 = -\frac{1}{|x|^2}\cdot\frac{x_1}{|x|}, \tag{11}$$

and so

$$\mathrm{grad}\,\frac{1}{|x|} = -\frac{1}{|x|^2}\cdot\frac{x}{|x|}, \qquad \text{or} \qquad \mathrm{grad}\,\frac{1}{|x-y|} = -\frac{1}{|x-y|^2}\cdot\frac{x-y}{|x-y|}, \tag{12}$$

for a fixed $y \in \mathbb{R}^3$. This means that if u is given by (9), then

$$\mathrm{grad}\, u(x) = \sum_{i=1}^m \mathrm{grad}\,\frac{Cq_i}{|x-y_i|} = -\sum_{i=1}^m \frac{Cq_i}{|x-y_i|^2}\cdot\frac{x-y_i}{|x-y_i|} = -E(x), \tag{13}$$

confirming (4). The case (10) can be treated similarly.

Next, we show that E is divergence-free in free space. When E is generated by a single particle at the origin, we have

$$\frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2} x_1 = -\frac{3}{2}\left(|x|^2\right)^{-5/2}\cdot 2x_1\cdot x_1 + \left(|x|^2\right)^{-3/2} = \frac{-2x_1^2+x_2^2+x_3^2}{|x|^5}, \tag{14}$$

and so

$$\begin{aligned}\mathrm{div}\frac{x}{|x|^3} &= \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2} x_1 + \frac{\partial}{\partial x_2}\left(|x|^2\right)^{-3/2} x_2 + \frac{\partial}{\partial x_3}\left(|x|^2\right)^{-3/2} x_3 \\ &= \frac{-2x_1^2+x_2^2+x_3^2}{|x|^5} + \frac{-2x_2^2+x_1^2+x_3^2}{|x|^5} + \frac{-2x_3^2+x_1^2+x_2^2}{|x|^5} = 0,\end{aligned}$$

for $x \neq 0$. Therefore, for either (2) or (3), we have

$$\mathrm{div} E = 0 \qquad \text{in free space.} \tag{15}$$

If we combine this with $E = -\mathrm{grad}\, u$, we get the Laplace equation

$$\Delta u = 0 \qquad \text{in free space.} \tag{16}$$

To proceed further, let us recall the *divergence theorem*, which asserts that

$$\int_U \mathrm{div} V = \int_{\partial U} V\cdot n, \tag{17}$$

where $U \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $V : U \to \mathbb{R}^3$ is a 3-dimensional vector field, and n is the outward pointing unit normal to the boundary ∂U. This theorem first appeared in Lagrange's 1760 work, and was proved in a special case by Gauss in 1813. The general 3-dimensional case was treated by Mikhail Vasilievich Ostrogradsky in 1826.

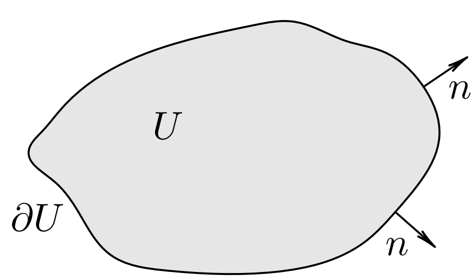


FIGURE 1. The setting of the divergence theorem.

If $V = \operatorname{grad} u$ then

$$\operatorname{div} V = \Delta u, \qquad \text{and} \qquad V \cdot n = n \cdot \operatorname{grad} u = \partial_n u, \tag{18}$$

where $\partial_n u$ is the normal derivative of u at ∂U, and so the divergence theorem yields

$$\int_U \Delta u = \int_{\partial U} \partial_n u \tag{19}$$

Now consider the potential generated by a point mass q at some point $y \in U$:

$$u(x) = \frac{Cq}{|x-y|}. \tag{20}$$

Let $B_\varepsilon = \{x \in \mathbb{R}^3 : |x-y| < \varepsilon\}$, and let us apply (19) to the domain $U_\varepsilon = U \setminus B_\varepsilon$. Since u is harmonic in U_ε, we infer

$$0 = \int_{U_\varepsilon} \Delta u = \int_{\partial U_\varepsilon} \partial_n u = \int_{\partial U} \partial_n u - \int_{\partial B_\varepsilon} \frac{\partial u}{\partial r}, \tag{21}$$

where $r = |x-y|$ is the radial variable centred at y. We can compute

$$\int_{\partial B_\varepsilon} \frac{\partial u}{\partial r} = \int_{\{r=\varepsilon\}} \frac{\partial}{\partial r}\frac{Cq}{r} = -\int_{\{r=\varepsilon\}} \frac{Cq}{r^2} = -4\pi\varepsilon^2 \frac{Cq}{\varepsilon^2} = -4\pi Cq, \tag{22}$$

and thus

$$\int_{\partial\Omega} \partial_n u = -4\pi Cq. \tag{23}$$

This is for the potential generated by a charge at $y \in U$. On the other hand, if $y \notin U$, then $\Delta u = 0$ in U, and hence we have

$$\int_{\partial U} \partial_n u = \int_U \Delta u = 0. \tag{24}$$

Therefore, for the potential generated by a set of point charges (9), we conclude that

$$\int_{\partial U} \partial_n u = -4\pi C \sum_{\{i : y_i \in U\}} q_i, \tag{25}$$

where the sum is taken over the indices i such that $y_i \in U$, meaning that the sum $\sum_{\{i:y_i\in U\}} q_i$ is simply the total charge contained in the domain U. Similarly, for the potential generated by a continuous distribution of charge (10), we get

$$\int_{\partial U} \partial_n u = -4\pi C \int_U \rho. \tag{26}$$

Finally, let $B_\varepsilon = \{y \in \mathbb{R}^3 : |y-x| < \varepsilon\}$ for $\varepsilon > 0$ small, and note that for any continuous function f, one has

$$f(x) = \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} f, \tag{27}$$

where $|B_\varepsilon|$ is the volume of the ball B_ε. By applying this result to Δu, with u given by (10), we infer

$$\Delta u(x) \approx \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \Delta u = \frac{1}{|B_\varepsilon|} \int_{\partial B_\varepsilon} \partial_n u = -\frac{4\pi C}{|B_\varepsilon|} \int_{B_\varepsilon} \rho \approx -4\pi C \rho(x), \tag{28}$$

where we used the identity (19) in the second step, and (26) in the third step. Upon taking the limit $\varepsilon \to 0$, we get

$$\Delta u(x) = -4\pi C \rho(x), \tag{29}$$

which is Poisson's equation.

3. Multipole expansions

In this section, we want to restrict ourselves to the *two dimensional* situation, which can be thought of as a stepping stone to understanding the full three dimensional setting. The main equations of two dimensional electrostatics and gravitation can be derived by assuming that the field potential $u(x) = u(x_1, x_2, x_3)$ does not depend on x_3. Since $\partial^2 u/\partial x_3^2 = 0$, the three dimensional Laplacian reduces to the two dimensional Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{30}$$

where we have used (x, y) instead of (x_1, x_2). Thus in free space, we have

$$\Delta u = 0. \tag{31}$$

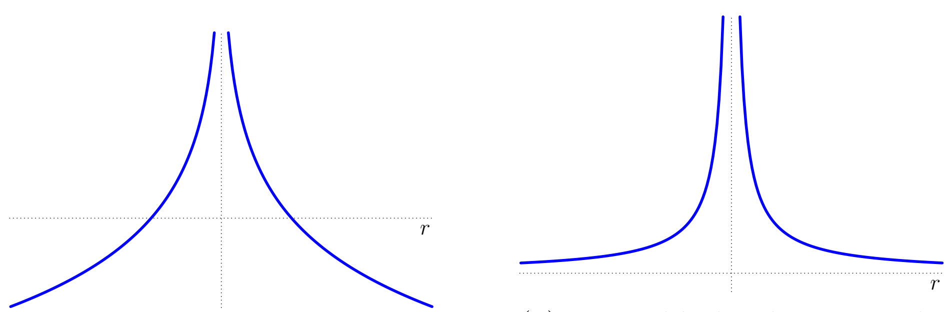
The potential generated by a point charge at $(0, 0)$ in 2 dimensions (or equivalently, the potential of a uniformly charged line coinciding with the z-axis in three dimensions) must clearly be radial, meaning that it must depend only on $r = \sqrt{x^2 + y^2}$. We know that the only radial solutions of $\Delta u = 0$ in 2 dimensions is

$$u(r, \theta) = A + B \log r. \tag{32}$$

Since the constant A has no effect on the field strength $E = -\operatorname{grad} u$, for simplicity, we pick $A = 0$. Then the potential generated by a point charge q at the origin is given by

$$u(r, \theta) = -Cq \log r = Cq \log \frac{1}{r}, \tag{33}$$

where the sign convention for the constant C is chosen so that if $Cq > 0$, then $E = -\operatorname{grad} u$ points away from the origin. This is the Newton/Coulomb law in 2 dimensions.



(A) In 2 dimensions, the potential generated by a point charge at the origin is unbounded both as $|x| \to \infty$, and as $x \to 0$.

(B) In 3 and higher dimensions, the potential generated by a point charge at the origin is unbounded as $x \to 0$, but tends to 0 as $|x| \to \infty$.

Figure 2. The potential generated by a point charge in different dimensions.

Now by invoking the identity (19) on the disk $D_\varepsilon = \{(x, y) : x^2 + y^2 < \varepsilon^2\}$, we get

$$\int_{D_\varepsilon} \Delta u = \int_{\partial D_\varepsilon} \frac{\partial u}{\partial r} = Cq \int_{\{r=\varepsilon\}} \frac{\partial}{\partial r} \log \frac{1}{r} = -\int_{\{r=\varepsilon\}} \frac{Cq}{r} = -2\pi\varepsilon \frac{Cq}{\varepsilon} = -2\pi Cq, \tag{34}$$

and proceeding as in the 3 dimensional case, we derive the Poisson equation

$$\Delta u = -2\pi C \mu \tag{35}$$

where μ is the charge (or mass) density in 2 dimensions.

The main purpose of this section is to compute the potential generated by a compact body located near the origin, in the form of a series. Suppose that $\mu = 0$ outside the disk D_R of radius R centred at the origin. Then $\Delta u = 0$ outside D_R, and hence u can be written as

$$u(r,\theta) = a_0 \log\frac{1}{r} + \sum_{n=1}^{\infty} \frac{a_n\cos(n\theta) + b_n \sin(n\theta)}{r^n} \tag{36}$$

There are no terms proportional to r^n in the expansion, because the term $\log\frac{1}{r}$ must dominate for large r. This expansion is known as the *multipole expansion* of harmonic potential functions in 2 dimensions. The question is now how to compute the coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ in terms of the density μ.

To get some initial insight, let us compute the coefficient a_0 first. For large r, the expansion (36) becomes $a_0 \log\frac{1}{r}$, so we expect that a_0 must be equal to the total charge of the body. To confirm this expectation, we start with the Poisson equation (35) and invoke the identity (19), yielding

$$-2\pi C\int_{\mathbb{R}^2}\mu = -2\pi C\int_{D_R}\mu = \int_{D_R}\Delta u = \int_{\partial D_R}\frac{\partial u}{\partial r}. \tag{37}$$

Then from (36) we compute

$$\frac{\partial u}{\partial r} = -\frac{a_0}{r} - \sum_{n=1}^{\infty} n\frac{a_n\cos(n\theta) + b_n\sin(n\theta)}{r^{n+1}}, \tag{38}$$

and integrate it over ∂D_R, to get

$$\int_{\partial D_R}\frac{\partial u}{\partial r} = \int_0^{2\pi}\frac{\partial u}{\partial r}(R,\theta)R\,\mathrm{d}\theta = -\frac{a_0}{R}\cdot 2\pi R = -2\pi a_0. \tag{39}$$

By comparing this with (37), we conclude

$$a_0 = C\int_{\mathbb{R}^2}\mu \tag{40}$$

which is the total charge/mass, up to the constant C.

In order to derive formulas for the remaining coefficients, we need some preparations. Let v be a sufficiently smooth scalar function. Then by applying the divergence theorem to the vector field $V = u \operatorname{grad} v$, we get *Green's first identity*

$$\int_{D_R}\operatorname{grad} u\cdot\operatorname{grad} v + \int_{D_R} u\Delta v = \int_{\partial D_R} u\frac{\partial v}{\partial r}. \tag{41}$$

Interchanging the roles of u and v in this identity, and subtracting the resulting identity from (41), we infer *Green's second identity*

$$\int_{D_R} u\Delta v - \int_{D_R} v\Delta u = \int_{\partial D_R} u\frac{\partial v}{\partial r} - \int_{\partial D_R} v\frac{\partial u}{\partial r}. \tag{42}$$

Note that (19) follows from (41) by putting $v \equiv 1$. The identities (41) and (42) can be considered as instances of, and are often called, *integration by parts* in higher dimensions.

Now the idea is to put a suitably chosen function v with $\Delta v = 0$ into (42). Then the left hand side becomes essentially an integral of $v\mu$, and the right hand side can be computed from (36). In other words, we use (42) as an extension of (37). A general harmonic function in D_R can be written as

$$v(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n\cos(n\theta) + B_n\sin(n\theta)). \tag{43}$$

We consider the simple choices $v(r,\theta) = r^n \cos(n\theta)$ and $v(r,\theta) = r^n \sin(n\theta)$, and compute the terms in the right hand side of (42), as follows. For $v(r,\theta) = r^n \cos(n\theta)$, we have

$$\begin{aligned}\int_{\partial D_R} u\frac{\partial v}{\partial r} &= \int_0^{2\pi} u(R,\theta)\frac{\partial v}{\partial r}(R,\theta)R\mathrm{d}\theta \\ &= \int_0^{2\pi}\left(a_0\log\frac{1}{R} + \sum_{n=1}^{\infty}\frac{a_n\cos(n\theta)+b_n\sin(n\theta)}{R^n}\right)\cdot nR^{n-1}\cos(n\theta)\cdot R\mathrm{d}\theta \\ &= \pi n a_n,\end{aligned}$$

and

$$\begin{aligned}\int_{\partial D_R} v\frac{\partial u}{\partial r} &= \int_0^{2\pi} v(R,\theta)\frac{\partial u}{\partial r}(R,\theta)R\mathrm{d}\theta \\ &= \int_0^{2\pi} R^n\cos(n\theta)\cdot\left(-\frac{a_0}{R} - \sum_{n=1}^{\infty} n\frac{a_n\cos(n\theta)+b_n\sin(n\theta)}{R^{n+1}}\right)\cdot R\mathrm{d}\theta \\ &= -\pi n a_n.\end{aligned}$$

Plugging this into (42), we get

$$2\pi n a_n = \int_{\partial D_R} u\frac{\partial v}{\partial r} - \int_{\partial D_R} v\frac{\partial u}{\partial r} = -\int_{D_R} v\Delta u = 2\pi C\int_{\mathbb{R}^2} v\mu \tag{44}$$

implying that

$$a_n = \frac{C}{n}\int_{\mathbb{R}^2} v\mu = \frac{C}{n}\int_{\mathbb{R}^2} r^n\cos(n\theta)\mu \tag{45}$$

Similarly, for $v(r,\theta) = r^n \sin(n\theta)$, we can compute

$$\int_{\partial D_R} u\frac{\partial v}{\partial r} = \pi n b_n, \quad \text{and} \qquad \int_{\partial D_R} v\frac{\partial u}{\partial r} = -\pi n b_n, \tag{46}$$

leading to the formula

$$b_n = \frac{C}{n}\int_{\mathbb{R}^2} v\mu = \frac{C}{n}\int_{\mathbb{R}^2} r^n\sin(n\theta)\mu \tag{47}$$

The formulas (40), (45), and (47) provide a complete prescription to compute the coefficients of the multipole expansion (36) in terms of the density μ.

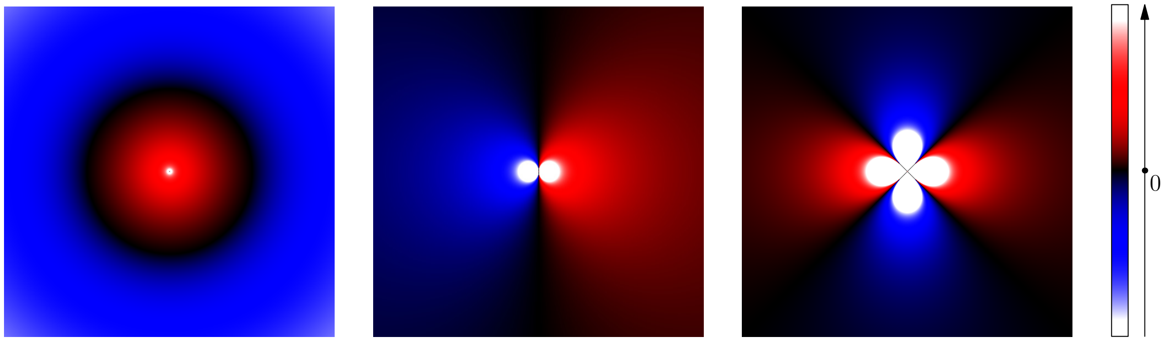
Definition 1. The number a_0 in the multipole expansion (36) is called the *monopole moment*. The vectors (a_1, b_1) and (a_2, b_2) are called the *dipole moment* and the *quadrupole moment*, respectively.

Figure 3 illustrates the first 3 terms of the multipole expansion

$$u(r,\theta) = \underbrace{a_0\log\frac{1}{r}}_{\text{monopole term}} + \underbrace{\frac{a_1\cos\theta + b_1\sin\theta}{r}}_{\text{dipole term}} + \underbrace{\frac{a_2\cos 2\theta + b_2\sin 2\theta}{r^2}}_{\text{quadrupole term}} + \ldots. \tag{48}$$

We depicted the dipole and quadrupole potentials with $(a_1, b_1) = (1, 0)$ and $(a_2, b_2) = (1, 0)$, respectively. A dipole (or quadrupole) potential with an arbitrary moment (a_1, b_1) (or (a_2, b_2)) is simply a rotation and scaling of the case depicted.

In the following examples, we are going to set $C = 1$, that is, we choose the normalization that the potential of a unit charge at the origin is given by $\log\frac{1}{r}$, cf. (33).



(A) Monopole potential (B) Dipole potential (C) Quadrupole potential

FIGURE 3. The first 3 terms of the multipole expansion (up to rotation).

Example 1. Suppose that the unit disk $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$ is uniformly charged with total charge 1 (Case 1, Figure 4). Let us compute the electrostatic potential generated by the disk. It is immediate that $\mu = \frac{1}{\pi}$ and $a_0 = 1$. Furthermore, we have

$$a_1 = \int_0^1 \int_0^{2\pi} r\cos\theta \cdot \frac{1}{\pi} \cdot r\mathrm{d}\theta\mathrm{d}r = \frac{1}{\pi}\int_0^1 r^2\mathrm{d}r \int_0^{2\pi} \cos\theta\mathrm{d}\theta = 0, \tag{49}$$

and

$$b_1 = \int_0^1 \int_0^{2\pi} r\sin\theta \cdot \frac{1}{\pi} \cdot r\mathrm{d}\theta\mathrm{d}r = \frac{1}{\pi}\int_0^1 r^2\mathrm{d}r \int_0^{2\pi} \sin\theta\mathrm{d}\theta = 0. \tag{50}$$

In fact, basically the same computation shows that $a_n = b_n = 0$ for all $n = 1, 2, \ldots$, and hence

$$u(r, \theta) = \log\frac{1}{r}. \tag{51}$$

We conclude that the electric field generated by a uniformly charged disk, measured outside the disk, is identical to the field generated by a point charge.

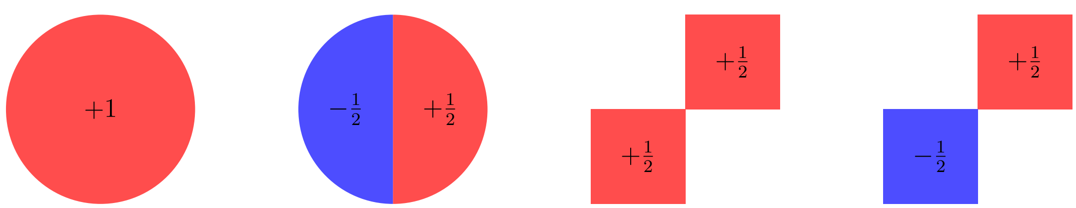


FIGURE 4. Examples of charged bodies.

Example 2. Now suppose that a half of the unit disk $\mathbb{D}$ carries a charge of $+\frac{1}{2}$, and the other half carries a charge of $-\frac{1}{2}$. More precisely, suppose that the half disk $\{x > 0\} \cap \mathbb{D}$ is uniformly charged with charge density $+\frac{1}{\pi}$, and the half disk $\{x < 0\} \cap \mathbb{D}$ is uniformly charged with charge density $-\frac{1}{\pi}$ (Case 2, Figure 4). Obviously, we have $a_0 = 0$. Then for a_1, we have

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^1 \int_{-\pi/2}^{\pi/2} r\cos\theta \cdot r\mathrm{d}\theta\mathrm{d}r - \frac{1}{\pi}\int_0^1 \int_{\pi/2}^{3\pi/2} r\cos\theta \cdot r\mathrm{d}\theta\mathrm{d}r \\ &= \frac{1}{\pi}\int_0^1 \left(\sin\theta\Big|_{-\pi/2}^{\pi/2}\right) r^2\mathrm{d}r - \frac{1}{\pi}\int_0^1 \left(\sin\theta\Big|_{\pi/2}^{3\pi/2}\right) r^2\mathrm{d}r = \frac{4}{3\pi}. \end{aligned}$$

Furthermore, we get

$$b_1 = \frac{1}{\pi}\int_0^1 \int_{-\pi/2}^{\pi/2} r\sin\theta \cdot r\mathrm{d}\theta\mathrm{d}r - \frac{1}{\pi}\int_0^1 \int_{\pi/2}^{3\pi/2} r\sin\theta \cdot r\mathrm{d}\theta\mathrm{d}r$$
$$= \frac{1}{\pi}\int_0^1 \left(-\cos\theta\Big|_{-\pi/2}^{\pi/2}\right) r^2\mathrm{d}r - \frac{1}{\pi}\int_0^1 \left(-\cos\theta\Big|_{\pi/2}^{3\pi/2}\right) r^2\mathrm{d}r = 0.$$

In fact, it can be shown that $a_n = b_n = 0$ for all $n = 2, 3, \ldots$, and hence

$$u(r,\theta) = \frac{4\cos\theta}{3\pi r}. \tag{52}$$

Example 3. Let the union of the two squares $Q = (0,1)^2$ and $\tilde{Q} = (-1,0)^2$ be uniformly charged with total charge $+1$ (Case 3, Figure 4). We have $\mu = \frac{1}{2}$ and $a_0 = 1$. Since $r\cos\theta = x$ and $r\sin\theta = y$, we have the general formulas

$$a_1 = C\int_{\mathbb{R}^2} x\mu(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{and} \qquad b_1 = C\int_{\mathbb{R}^2} y\mu(x,y)\,\mathrm{d}x\mathrm{d}y \tag{53}$$

Invoking these formulas, we infer

$$a_1 = \int_Q x\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} x\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 x\,\mathrm{d}x\int_0^1 \mathrm{d}y + \frac{1}{2}\int_{-1}^0 x\,\mathrm{d}x\int_{-1}^0 \mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 x\,\mathrm{d}x + \frac{1}{2}\int_{-1}^0 x\,\mathrm{d}x = \frac{1}{2}\int_{-1}^1 x\,\mathrm{d}x = 0,$$

and similarly,

$$b_1 = \int_Q y\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} y\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 y\,\mathrm{d}y\int_0^1 \mathrm{d}x + \frac{1}{2}\int_{-1}^0 y\,\mathrm{d}y\int_{-1}^0 \mathrm{d}x = 0.$$

As for the quadrupole moment, noting that

$$r^2\cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2, \tag{54}$$

and

$$r^2\sin 2\theta = 2r^2\sin\theta\cos\theta = 2xy, \tag{55}$$

we derive the general formulas

$$a_2 = \frac{C}{2}\int_{\mathbb{R}^2} (x^2 - y^2)\mu(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{and} \qquad b_2 = C\int_{\mathbb{R}^2} xy\mu(x,y)\,\mathrm{d}x\mathrm{d}y \tag{56}$$

With the help of these formulas, we compute

$$a_2 = \frac{1}{2}\int_Q (x^2 - y^2)\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} (x^2 - y^2)\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{4}\int_0^1\int_0^1 x^2\,\mathrm{d}x\mathrm{d}y - \frac{1}{4}\int_0^1\int_0^1 y^2\,\mathrm{d}x\mathrm{d}y + \frac{1}{4}\int_{-1}^0\int_{-1}^0 x^2\,\mathrm{d}x\mathrm{d}y - \frac{1}{4}\int_{-1}^0\int_{-1}^0 y^2\,\mathrm{d}x\mathrm{d}y = 0,$$

and

$$b_2 = \int_Q xy\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} xy\cdot\frac{1}{2}\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 x\,\mathrm{d}x\int_0^1 y\,\mathrm{d}y + \frac{1}{2}\int_{-1}^0 x\,\mathrm{d}x\int_{-1}^0 y\,\mathrm{d}y = \frac{1}{4}.$$

Finally, we conclude that the multipole expansion of the potential up to and including the quadrature term is

$$u(r,\theta) = \log\frac{1}{r} + \frac{\sin 2\theta}{4r^2} + \ldots = \log\frac{1}{r} + \frac{\sin\theta\cos\theta}{2r^2} + \ldots. \tag{57}$$

Example 4. Consider the situation where $Q = (0, 1)^2$ is uniformly charged with total charge $+\frac{1}{2}$, and $\tilde{Q} = (-1, 0)^2$ is uniformly charged with total charge $-\frac{1}{2}$ (Case 4, Figure 4). We have $\mu = \frac{1}{2}$ in Q, $\mu = -\frac{1}{2}$ in $\tilde{Q}$, and $a_0 = 0$. To compute the dipole moment, we use the formulas (53), and get

$$\begin{aligned} a_1 &= \int_Q x \cdot \frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} x \cdot \big(-\frac{1}{2}\big)\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 x\,\mathrm{d}x \int_0^1 \mathrm{d}y - \frac{1}{2}\int_{-1}^0 x\,\mathrm{d}x \int_{-1}^0 \mathrm{d}y \\ &= \frac{1}{2}\int_0^1 x\,\mathrm{d}x - \frac{1}{2}\int_{-1}^0 x\,\mathrm{d}x = \frac{1}{2}, \end{aligned}$$

and similarly,

$$b_1 = \int_Q y \cdot \frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} y \cdot \big(-\frac{1}{2}\big)\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 y\,\mathrm{d}y \int_0^1 \mathrm{d}x - \frac{1}{2}\int_{-1}^0 y\,\mathrm{d}y \int_{-1}^0 \mathrm{d}x = \frac{1}{2}.$$

As for the quadrupole moment, the formulas (56) yield

$$a_2 = \frac{1}{2}\int_Q (x^2 - y^2) \cdot \frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\tilde{Q}} (x^2 - y^2) \cdot \big(-\frac{1}{2}\big)\,\mathrm{d}x\mathrm{d}y = 0,$$

and

$$b_2 = \int_Q xy \cdot \frac{1}{2}\,\mathrm{d}x\mathrm{d}y + \int_{\tilde{Q}} xy \cdot \big(-\frac{1}{2}\big)\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 x\,\mathrm{d}x \int_0^1 y\,\mathrm{d}y - \frac{1}{2}\int_{-1}^0 x\,\mathrm{d}x \int_{-1}^0 y\,\mathrm{d}y = 0.$$

Therefore, the potential has the expansion

$$u(r, \theta) = \frac{\cos\theta + \sin\theta}{2r} + O\big(\frac{1}{r^3}\big), \tag{58}$$

where the error term $O(r^{-3})$ is to indicate that there is no quadrupole term in the expansion.