Problem 5.1:6

Statement. Consider the problem
\[
\begin{cases}
  u_{tt} = u_{xx}, & 0 \leq x \leq \pi, \quad -\infty < t < \infty, \\
  u(0, t) = 0, & u(\pi, t) = 0, \\
  u(x, 0) = x(\pi - x), & u_t(x, 0) = 0.
\end{cases}
\]

(a) Find a function that satisfies the equation and the boundary conditions exactly, and
the initial condition to within an error of .001.

(b) By computing \( u_{tt} \) and \( u_{xx} \) at \((x, t) = (0, 0)\), show that there is no \( C^2 \) solution of the problem.

Solution. (a) Let us first compute the sine series coefficients for \( f(x) = x(\pi - x) \), which are
given by
\[
b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx, \quad n = 1, 2, \ldots.
\]

We need the integrals
\[
\int_0^\pi x \sin nx \, dx = -\frac{x \cos nx}{n}\bigg|_0^\pi + \frac{1}{n} \int_0^\pi \cos nx \, dx = \frac{(-1)^{n+1} \pi}{n},
\]
and
\[
\int_0^\pi x^2 \sin nx \, dx = -\frac{x^2 \cos nx}{n^2}\bigg|_0^\pi + \frac{2}{n} \int_0^\pi x \cos nx \, dx
\]
\[
= \frac{(-1)^{n+1} \pi^2}{n} + \frac{2}{n^2} x \sin nx\bigg|_0^\pi - \frac{2}{n^2} \int_0^\pi \sin nx \, dx
\]
\[
= \frac{(-1)^{n+1} \pi^2}{n} + \frac{2 \cos nx}{n^2}\bigg|_0^\pi = \frac{(-1)^{n+1} \pi^2}{n} + 2((-1)^n - 1).
\]

Hence
\[
b_n = (-1)^{n+1} \frac{2\pi}{n} - (-1)^{n+1} \frac{2\pi}{n} + 4(1 - (-1)^n) = \frac{4(1 - (-1)^n)}{n^2 \pi},
\]
and the sine series for \( f \) is
\[
f(x) = \frac{8}{\pi} \sum_{m=0}^\infty \frac{\sin(2m+1)x}{(2m+1)^2}.\]

Obviously, for any \( M > 0 \), the function
\[
u_M(x, t) = \frac{8}{\pi} \sum_{m=0}^M \frac{\sin(2m+1)x}{(2m+1)^2} \cos(2m+1)t,
\]

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satisfies the wave equation $u_{tt} = u_{xx}$ and the boundary conditions $u(0, t) = u(\pi, t) = 0$. We just need to choose $M$ so large that $|u_M(x, 0) - f(x)| \leq .001$ for all $0 \leq x \leq \pi$. We can estimate this error as follows:

$$|f(x) - u_M(x, 0)| \leq \frac{8}{\pi} \left| \sum_{m=M+1}^{\infty} \frac{\sin((2m+1)x)}{(2m+1)^2} \right| \leq \frac{8}{\pi} \sum_{m=M+1}^{\infty} \frac{1}{(2m+1)^2}$$

$$\leq \frac{4}{\pi} \int_{2M+1}^{\infty} \frac{d\theta}{(2M+1)\pi}.$$

Therefore, to ensure $|u_M(x, 0) - f(x)| \leq .001$, it is sufficient to take $M \geq \frac{2000}{\pi} - \frac{1}{2}$, i.e., $M \geq 637$.

**Problem 5.1:9**

**Statement.** Use separation of variables to find all product solutions of the problem

$$\begin{cases} u_{tt} = a^2 u_{xx} - ku_t, & 0 \leq x \leq L, \quad -\infty < t < \infty, \\ u(0, t) = 0, & u(L, t) = 0, \end{cases}$$

for the string with air resistance and fixed ends (assume $k > 0$).

**Solution.** Putting $u(x, t) = X(x)T(t)$, we have

$$XT'' = a^2 X''T - kXT',$$

and division by $XT$ gives

$$\frac{T''}{T} + k \frac{T'}{T} = a^2 \frac{X''}{X} = a^2 \alpha,$$

where $\alpha$ is a constant. If $\alpha = 0$, then the equation for $X$ becomes $X'' = 0$, meaning that $X(x) = Ax + B$. But the boundary conditions $X(0) = X(L) = 0$ forces $X(x) \equiv 0$. So this case is trivial. Now if $\alpha = \lambda^2 > 0$ with $\lambda > 0$, we have

$$X(x) = A e^{\lambda x} + B e^{-\lambda x}.$$

The boundary conditions give $A + B = 0$ and $A e^\lambda + B e^{-\lambda} = 0$, implying that $A = B = 0$. Finally, consider the remaining case $\alpha = -\lambda^2 < 0$ with $\lambda > 0$. The general solution for $X$ is

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x),$$

and from $X(0) = 0$ we immediately get $A = 0$. Then $X(L) = B \sin(\lambda L) = 0$ gives the condition $\lambda = \frac{n\pi}{L}$ for some positive integer $n$. To conclude the analysis of the equation for $X$, the only solutions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots,$$

and their linear combinations.

We shall consider the equation for $T$. With $\alpha = -\lambda^2 = -\frac{n^2\pi^2}{L^2}$, we have

$$T'' + kT' + \omega_n^2 T = 0,$$

where $\omega_n = \frac{n\pi}{L}$. This (standard equation for damped oscillator) can easily be solved by the ansatz $T(t) = e^{\mu t}$, which yields

$$\mu = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \omega_n^2}.$$

If $\omega_n < \frac{k}{2}$, we have two monotone solutions

$$T_n(t) = Ae^{-\omega_n t} + Be^{\omega_n t}, \quad k_n = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \omega_n^2}.$$
If \( \omega_n > \frac{k}{2} \), we have the oscillating solutions

\[
T_n(t) = e^{-kt/2}(A \cos \tilde{\omega}_n t + B \sin \tilde{\omega}_n t), \quad \tilde{\omega}_n = \sqrt{\omega_n^2 - \frac{k^2}{4}}.
\]

If it so happens that \( \omega_n = \frac{k}{2} \), we have

\[
T_n(t) = e^{-kt/2}(A + Bt).
\]

To conclude, all product solutions of the given problem are given by

\[
u(x,t) = T_n(x) \sin \left( \frac{n \pi x}{L} \right),
\]

as \( n \) ranges over the positive integers, where \( T_n \) is one of the above three functions depending on how \( \omega_n = \frac{n \pi a}{L} \) compares with \( \frac{k}{2} \). Note that given \( n \), \( T_n \) is one and only one of the above three choices.

**Problem 5.2:1**

**Statement.** Find the solution of

\[
\begin{aligned}
&u_{tt} = a^2 u_{xx}, \quad x, t \in \mathbb{R}, \\
u(x,0) = f(x), \quad u_t(x,0) = g(x),
\end{aligned}
\]

in the following cases:

(a) \( f(x) = e^{-x^2}, \, g(x) = 2axe^{-x^2} \),

(b) \( f(x) = 1, \, g(x) = 0 \),

(d) \( f(x) = 0, \, g(x) = \sin^2 x \).

**Solution.** We can directly apply D’Alambert’s formula

\[
u(x,t) = \frac{f(x+at) + f(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds.
\]

(b) We have

\[
\int 2xe^{-x^2} dx = -e^{-x^2} + C,
\]

hence

\[
u(x,t) = \frac{e^{-(x+at)^2} + e^{-(x-at)^2}}{2} + \frac{1}{2a} \int_{x-at}^{x+at} 2ase^{-s^2} ds
\]

\[
= \frac{e^{-(x+at)^2} + e^{-(x-at)^2}}{2} + \frac{1}{2} \left( e^{-(x+at)^2} + e^{-(x-at)^2} \right)
\]

\[
= e^{-(x-at)^2}.
\]

(d) \( u(x,t) = 1 \).

(f) We have

\[
u(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} \sin^2 s ds = \frac{1}{4a} \int_{x-at}^{x+at} (1 - \cos 2s) ds = \frac{1}{2} t + \frac{\sin 2(x - at) - \sin 2(x + at)}{8a}.
\]

**Problem 5.2:4**

**Statement.** Solve

\[
\begin{aligned}
&u_{tt} = a^2 u_{xx}, \quad 0 < x < \infty, \quad -\infty < t < \infty, \\
u_x(0,t) = 0, \\
u(x,0) = x^3, \quad u_t(x,0) = 0.
\end{aligned}
\]
Solution. We need first to extend the initial data by even reflection to the entire real line $-\infty < c < \infty$, apply D’Alambert’s formula to solve the problem on the line, and finally restrict to the half line $x \geq 0$. The even reflection of the initial datum $x^3$ is $|x|^3$. Let us apply the D’Alambert formula

$$u(x, t) = \frac{|x - at|^3 + |x + at|^3}{2}.$$  

Note that since $f(x) = |x|^3$ is a $C^2$ function, $u(x, t)$ satisfies the wave equation for all $x$ and $t$. We can calculate

$$u_x(x, t) = \frac{3(x - at)|x - at| + 3(x + at)|x + at|}{2},$$

which implies that $u_x(0, t) = 0$ for all $t$. To solve the original problem, we just need to restrict our attention to the region $x \geq 0$. 
