# SOLUTIONS TO PROBLEMS FROM ASSIGNMENT 6

## Problems 4.1:6

**Statement.** Assuming $\operatorname{FS} f(x) = f(x) = x^2$ for $-L \le x \le L$, obtain the results

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}, \tag{1}$$

and

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}. \tag{2}$$

From these results, obtain

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8},$$

and

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots = \frac{\pi^2}{24}.$$

**Solution.** For convenience, let us take $L = \pi$. Since the function $f(x) = x^2$ is even in $[-\pi, \pi]$, the Fourier series of $f$ involve only cosines. It is given by

$$\operatorname{FS} f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad \text{with} \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, \mathrm{d}x.$$

For $n = 0$, we have

$$a_0 = \frac{2}{\pi} \int_0^\pi x^2 \, \mathrm{d}x = \frac{2\pi^2}{3}.$$

Now assuming $n > 0$, let us compute the integral

$$\begin{aligned} \int_0^\pi x^2 \cos nx \, \mathrm{d}x &= \frac{x^2 \sin nx}{n} \Big|_0^\pi - \frac{2}{n} \int_0^\pi x \sin nx \, \mathrm{d}x \\ &= 0 + \frac{2x \cos nx}{n^2} \Big|_0^\pi - \frac{2}{n^2} \int_0^\pi \cos nx \, \mathrm{d}x = \frac{2\pi(-1)^n}{n^2}, \end{aligned}$$

which implies $a_n = \frac{4(-1)^n}{n^2}$. We conclude that

$$x^2 = \operatorname{FS} x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx = \frac{\pi^2}{3} + 4 \left( -\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots \right).$$

Putting $x = 0$, we get

$$0 = \frac{\pi^2}{3} + 4 \left( -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots \right),$$

hence (1). On the other hand, for $x = \pi$ we get

$$\pi^2 = \frac{\pi^2}{3} + 4 \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right),$$

giving (2). The other two identities can be derived by taking the sum and the difference of (1) and (2).

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## Problem 4.2:6

**Statement.** Find a sequence of functions $f_1, f_2, \ldots$ defined on $[-L, L]$, such that for each $x$ in $[-L, L]$, $\lim_{n\to\infty} f_n(x) = 0$ and yet

$$\lim_{n\to\infty} \int_{-L}^{L} f_n(x)\mathrm{d}x \neq \int_{-L}^{L} \Big(\lim_{n\to\infty} f_n(x)\Big)\,\mathrm{d}x.$$

**Solution.** Define the function $g(x)$ for $-\infty < x < \infty$ by

$$g(x) = \begin{cases} x, & \text{if} \quad 0 \le x \le 1, \\ 2 - x, & \text{if} \quad 1 \le x \le 2, \\ 0, & \text{otherwise}. \end{cases}$$

Note that the integral of $g$ is 1, and $g$ vanishes outside the interval $(0, 2)$. Now define

$$f_n(x) = ng(nx), \qquad \text{for} \quad n = 1, 2, \ldots.$$

We have

$$\int_{-L}^{L} f_n(x)\,\mathrm{d}x = \int_{-L}^{L} ng(nx)\,\mathrm{d}x = \int_{-nL}^{nL} g(y)\,\mathrm{d}y = 1,$$

for all large $n$ (specifically, if $nL \ge 2$). Also, $f_n(x) = 0$ if $x \le 0$ or $x \ge \frac{2}{n}$. Hence $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [-L, L]$. To see clearly the convergence $f_n(x) \to 0$ for $x > 0$, observe that given any $x^* > 0$, $f_n(x^*) = 0$ if $n \ge \frac{2}{x^*}$. Note also that we could have started with any $g$ with nonzero integral satisfying $g(0) = 0$ and $g(x) \to 0$ as $x \to \pm\infty$.

## Problem 4.2:15

**Statement.** It can be shown that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$$

converges for all $x$ and that it is an absoletely integrable function of $x$ on the interval $[-\pi, \pi]$. Use Bessel's inequality to show that there is no function $f(x)$ defined on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} f(x)^2\mathrm{d}x < \infty$, such that

$$\mathrm{FS}\, f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}.$$

**Solution.** In this setting, if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

then Bessel's inequality holds:

$$\sum_{n=1}^{N} b_n^2 \le \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\mathrm{d}x,$$

for any $N$. Since $b_n = \frac{1}{\sqrt{n}}$, the sum on left hand side diverges to $\infty$ as $N \to \infty$, which shows that the integral on the right hand side cannot be finite.

## Problem 4.3:9

**Statement.** (a) Find a formal solution of the problem

$$\begin{cases} u_t = ku_{xx}, & 0 \le x \le 1,\ t \ge 0, \\ u(0,t) = 0, & u(1,t) = 0, \\ u(x,0) = f(x), \end{cases}$$

where

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

(b) If $u_t(x,t)$ is formally computed by differentiating each term of the formal solution with respect to $t$, then show that $u_t(\frac{1}{2},0) = -\infty$ results. Provide a physical explanation of this observation by considering the flux of heat through the ends of a small interval centred at $x = \frac{1}{2}$.

**Solution.** (a) The formal solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n\pi^2 kt} \sin(n\pi x), \qquad \text{with} \quad b_n = 2\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x.$$

Note that $\sin(n\pi(1-x)) = \sin(n\pi - n\pi x) = \sin(n\pi x)$ when $n$ is odd, and $\sin(n\pi - n\pi x) = -\sin(n\pi x)$ when $n$ is even. Using this symmetry, and taking into account that $f(1-x) = f(x)$, we infer

$$b_n = 2\int_0^{\frac{1}{2}} f(x)\sin(n\pi x)\,\mathrm{d}x + 2(-1)^{n+1}\int_0^{\frac{1}{2}} f(x)\sin(n\pi x)\,\mathrm{d}x.$$

So $b_n = 0$ for even $n$, and

$$\begin{aligned} b_n &= 4\int_0^{\frac{1}{2}} f(x)\sin(n\pi x)\,\mathrm{d}x = -\frac{4}{n\pi}\cos(n\pi x)\Big|_0^{\frac{1}{2}} + \frac{4}{n\pi}\int_0^{\frac{1}{2}}\cos(n\pi x)\,\mathrm{d}x \\ &= \frac{4}{n\pi} + (-1)^m\frac{4}{n^2\pi^2}, \end{aligned}$$

for odd $n$, with $n = 2m+1$. Putting everything together, we conclude

$$u(x,t) = \sum_{m=0}^{\infty}\left(\frac{4}{n\pi} + (-1)^m\frac{4}{n^2\pi^2}\right)e^{-n\pi^2 kt}\sin(n\pi x),$$

where $n$ depends on $m$ as $n = 2m+1$.

(b) Formally, we compute

$$u_t(x,t) = -\sum_{m=0}^{\infty}\left(4\pi k + (-1)^m\frac{4k}{n}\right)e^{-n\pi^2 kt}\sin(n\pi x),$$

and so

$$\begin{aligned} u_t(\tfrac{1}{2},0) &= -4k\sum_{m=0}^{\infty}\left(\pi + \frac{(-1)^m}{2m+1}\right)\sin\frac{(2m+1)\pi}{2} \\ &= -4k\sum_{m=0}^{\infty}\left((-1)^m\pi + \frac{1}{2m+1}\right) = -\infty, \end{aligned}$$

since $k > 0$ and $\sum_{m=0}^{\infty}\frac{1}{2m+1} = \infty$. This result suggests that the temperature at the point $x = \frac{1}{2}$ drops infinitely fast for a very short (in fact infinitesimal) time near $t = 0$.

As suggested in the statement, we can also formally compute the flux as

$$\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} u_t(x,0)\,\mathrm{d}x = k\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} u_{xx}(x,0)\,\mathrm{d}x = ku_x(x,0)\Big|_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} = -2k,$$

where $\varepsilon > 0$ is small. We see that no matter how small the interval $[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$ is, the integral of $u_t$ over it is a fixed negative number. Therefore, the function $u_t$ must become negative infinity at $x = \frac{1}{2}$.

## Problem 4.3:12

**Statement.** Find a formal solution of the problem

$$\begin{cases} u_t = ku_{xx}, \quad 0 \le x \le 10,\, t \ge 0, \\ u_x(0,t) = 2, \quad u_x(10,t) = 3, \\ u(x,0) = 0. \end{cases}$$

**Solution.** First of all, we need to shift the unknown function so that the boundary conditions are homogeneous. Looking for a polynomial $p(x) = Ax^2 + Bx$ satisfying $p'(0) = 2$ and $p'(10) = 3$, we find $A = \frac{1}{20}$ and $B = 2$. Now define the new unknown $v = u - p$, so that $u = p + v$. Then since $u_t = v_t$ and $u_{xx} = v_{xx} + 2A$, we see that $v$ must satisfy

$$\begin{cases} v_t = kv_{xx} + 2kA, \quad 0 \le x \le 10,\, t \ge 0, \\ v_x(0,t) = 0, \quad v_x(10,t) = 0, \\ v(x,0) = -p(x) = -Ax^2 - Bx. \end{cases}$$

In order to use separation of variables, we assume

$$v(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(\frac{n\pi x}{10}),$$

and formally substitute it into $v_t = kv_{xx} + 2kA$, to get

$$a_0'(t) = 2kA,$$

and

$$a_n'(t) = -\frac{n^2\pi^2 k}{100} a_n(t), \qquad n > 0.$$

Note that the cosine series of $2kA$ involves only the constant term, so it does not affect at all the equations for $n > 0$, which remain the same as the equation for the homogeneous case $v_t = kv_{xx}$. The equations are easily solved as

$$a_0(t) = a_0(0) + 2kAt, \qquad \text{and} \qquad a_n(t) = a_n(0)\exp(-\frac{n^2\pi^2 k}{100}t), \quad n > 0.$$

Obviously, $a_n(0)$ for $n \ge 0$ are the cosine series coefficients of the initial datum $v(x,0) = -p(x)$, which are given by

$$a_0(0) = -\frac{1}{10}\int_0^{10} p(x)\,\mathrm{d}x, \qquad \text{and} \qquad a_n(0) = -\frac{1}{5}\int_0^{10} p(x)\cos(\frac{n\pi x}{10})\,\mathrm{d}x, \quad n > 0.$$

Let us do the computation. We have

$$\int_0^{10} p(x)\,\mathrm{d}x = \left(\frac{Ax^3}{3} + \frac{Bx^2}{2}\right)\Big|_0^{10} = \frac{50}{3} + 100,$$

and

$$\begin{aligned}\int_0^{10} x\cos(\frac{n\pi x}{10})\,\mathrm{d}x &= \frac{10}{n\pi}x\sin(\frac{n\pi x}{10})\Big|_0^{10} - \frac{10}{n\pi}\int_0^{10}\sin(\frac{n\pi x}{10})\,\mathrm{d}x\\ &= \frac{100}{n^2\pi^2}\cos(\frac{n\pi x}{10})\Big|_0^{10} = ((-1)^n-1)\frac{100}{n^2\pi^2},\end{aligned}$$

as well as

$$\begin{aligned}\int_0^{10} x^2\cos(\frac{n\pi x}{10})\,\mathrm{d}x &= \frac{10}{n\pi}x^2\sin(\frac{n\pi x}{10})\Big|_0^{10} - \frac{20}{n\pi}\int_0^{10}x\sin(\frac{n\pi x}{10})\,\mathrm{d}x\\ &= \frac{200}{n^2\pi^2}x\cos(\frac{n\pi x}{10})\Big|_0^{10} - \frac{200}{n^2\pi^2}\int_0^{10}\cos(\frac{n\pi x}{10})\,\mathrm{d}x\\ &= (-1)^n\frac{2000}{n^2\pi^2},\end{aligned}$$

leading to

$$a_0(0) = \frac{35}{3}, \qquad a_n(0) = -(-1)^n\frac{400}{n^2\pi^2}A + (1-(-1)^n)\frac{20}{n^2\pi^2}B = \frac{40+60(-1)^{n+1}}{n^2\pi^2}.$$

Therefore, we conclude

$$v(x,t) = \frac{35}{3} + \frac{k}{10}t + \sum_{n=1}^{\infty}\frac{40+60(-1)^{n+1}}{n^2\pi^2}\exp(-\frac{n^2\pi^2k}{100}t)\cos(\frac{n\pi x}{10}),$$

and so

$$u(x,t) = \frac{35}{3} + 2x + \frac{1}{20}x^2 + \frac{k}{10}t + \sum_{n=1}^{\infty}\frac{40+60(-1)^{n+1}}{n^2\pi^2}\exp(-\frac{n^2\pi^2k}{100}t)\cos(\frac{n\pi x}{10}).$$

## Problem 5.1:1

**Statement.** Solve the problem

$$\begin{cases} u_{tt} = a^2u_{xx}, \quad 0\le x\le L,\ -\infty < t < \infty,\\ u(0,t) = u(L,t) = 0,\\ u(x,0) = f(x), \quad u_t(x,0) = g(x),\end{cases}$$

in the following cases

(a) $f(x) = 3\sin(\frac{\pi x}{L}) - \sin(\frac{4\pi x}{L})$, $g(x) = \frac{1}{2}\sin(\frac{2\pi x}{L})$,
(b) $f(x) = \sin^3(\frac{\pi x}{L})$, $g(x) = 0$,
(c) $f(x) = 0$, $g(x) = \sin(\frac{\pi x}{L})\cos^2(\frac{\pi x}{L})$,
(d) $f(x) = \sin^3(\frac{\pi x}{L})$, $g(x) = \sin(\frac{\pi x}{L})\cos^2(\frac{\pi x}{L})$.

**Solution.** If the initial data satisfy

$$f(x) = \sum_{n=1}^{N}\alpha_n\sin(\frac{n\pi x}{L}), \qquad g(x) = \sum_{n=1}^{N}\beta_n\sin(\frac{n\pi x}{L}),$$

then the solution is given by

$$u(x,t) = \sum_{n=1}^{N}\left(\alpha_n\cos(\frac{n\pi at}{L}) + \beta_n\frac{L}{n\pi a}\sin(\frac{n\pi at}{L})\right)\sin(\frac{n\pi x}{L}).$$

Applying this formula, we get the following.

(a) $u(x,t) = 3\cos(\frac{\pi at}{L})\sin(\frac{\pi x}{L}) - \cos(\frac{4\pi at}{L})\sin(\frac{4\pi x}{L}) + \frac{L}{4\pi a}\sin(\frac{2\pi at}{L})\sin(\frac{2\pi x}{L})$.

(b) From the triple angle formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we have $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$, hence $u(x,t) = \frac{3}{4}\cos(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) - \frac{1}{4}\cos(\frac{3\pi a t}{L})\sin(\frac{3\pi x}{L})$.

(c) Again using the triple angle formula $g(x) = \sin(\frac{\pi x}{L}) - \sin^3(\frac{\pi x}{L}) = \frac{1}{4}\sin(\frac{\pi x}{L}) + \frac{1}{4}\sin(\frac{3\pi x}{L})$, which leads to $u(x,t) = \frac{L}{4\pi a}\sin(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) + \frac{L}{12\pi a}\sin(\frac{3\pi a t}{L})\sin(\frac{3\pi x}{L})$.

(d) By combining (b) and (c) above, we infer

$$u(x,t) = \big(\tfrac{3}{4}\cos(\tfrac{\pi a t}{L}) + \tfrac{L}{4\pi a}\sin(\tfrac{\pi a t}{L})\big)\sin(\tfrac{\pi x}{L}) + \big(-\tfrac{1}{4}\cos(\tfrac{3\pi a t}{L}) + \tfrac{L}{12\pi a}\sin(\tfrac{3\pi a t}{L})\big)\sin(\tfrac{3\pi x}{L}).$$