SOLUTIONS TO PROBLEMS FROM ASSIGNMENT 5

Problems 3.1:6bd

Statement. Solve the problem

$$u_t = u_{xx}, \qquad (t \geq 0),$$

with the initial condition $u(x,0) = f(x)$, where the functions $u(x,t)$ and $f(x)$ are assumed to be 2π-periodic in the x variable. The function f is given in each case by

(b) $f(x) = \frac{1}{2} + \cos(2x) - 6\sin(2x)$,
(d) $f(x) = 6\sin(x) - 7\cos(3x) - 7\sin(3x)$.

Solution. We know that the solution of the above problem with the initial condition

$$f(x) = A_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)),$$

is given by

$$u(x,t) = A_0 + \sum_{n=1}^{N} e^{-n^2 t}(a_n \cos(nx) + b_n \sin(nx)).$$

A direct application of this formula gives

(b) $u(x,t) = \frac{1}{2} + e^{-4t}\cos(2x) - 6e^{-4t}\sin(2x)$,
(d) $u(x,t) = 6e^{-t}\sin(x) - 7e^{-9t}\cos(3x) - 7e^{-9t}\sin(3x)$.

Problem 3.1:8

Statement.

(a) Consider the problem

$$\begin{aligned} u_t &= k u_{xx}, \qquad (x \geq 0,\ t \geq 0),\\ u(0,t) &= \cos(\omega t), \qquad (t \geq 0). \end{aligned} \tag{1}$$

This is a heat conduction problem for a semi-infinite rod ($x \geq 0$) whose end (at $x = 0$) is subjected to a periodic temperature variation $u(0,t) = \cos(\omega t)$. Use the particular solutions

$$u(x,t) = Ae^{\lambda x}\cos(\lambda x + 2k\lambda^2 t) + Be^{\lambda x}\sin(\lambda x + 2k\lambda^2 t), \tag{2}$$

to find a solution of this problem which has both of the additional properties:

(P1) $u(x,t) \to 0$ as $x \to \infty$,
(P2) $u(x, t + \frac{2\pi}{\omega}) = u(x,t)$.

(b) Show that the solution of (1) is not unique, if either (P1) or (P2) is omitted.

(c) Assuming that $\omega = \frac{\pi}{2}$ and $k = \frac{\pi}{4}$, roughly sketch the graph of the temperature distribution in the xu-plane when $t = 0, 1, 2, 3, 4$, paying attention to where $u(x,t) = 0$.

(d) Show that at any fixed time t, the distance between consecutive local maxima, say x_1 and x_2, of $u(x,t)$ is $2\pi\sqrt{\frac{2k}{\omega}}$, and show that the ratio $u(x_2,t)/u(x_1,t)$ is $e^{-2\pi} \approx 0.00187$, regardless of the positive values of k and ω.

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Solution. (a) In view of (2), the boundary condition $u(0,t) = \cos(\omega t)$ gives

$$u(0,t) = A\cos(2k\lambda^2 t) + B\sin(2k\lambda^2 t) = \cos(\omega t), \qquad t \geq 0,$$

implying that $A = 1$, $B = 0$, and $\lambda = \pm\sqrt{\frac{\omega}{2k}}$, i.e., we have the solution

$$u(x,t) = \exp(\pm\sqrt{\tfrac{\omega}{2k}}x)\cos(\pm\sqrt{\tfrac{\omega}{2k}}x + \omega t). \tag{3}$$

In order to satisfy (P1) we need to choose the minus sign in $\pm\sqrt{\frac{\omega}{2k}}$, so we finally have

$$u(x,t) = \exp(-\sqrt{\tfrac{\omega}{2k}}x)\cos(-\sqrt{\tfrac{\omega}{2k}}x + \omega t). \tag{4}$$

It is clear that this solution satisfies (P2).

(b) If (P1) is omitted, we can choose either plus or minus sign in (3), which means that the solution is not unique. On the other hand, if (P2) is dropped, we can add any $v(x,t)$ satisfying

$$v_t = kv_{xx}, \qquad (x \geq 0,\, t \geq 0),$$
$$v(0,t) = 0, \qquad (t \geq 0),$$

to $u(x,t)$. For example, we can take

$$v(x,t) = \frac{1}{\sqrt{t+1}}\left(\exp\left(-\frac{(x-1)^2}{4k(t+1)}\right) - \exp\left(-\frac{(x+1)^2}{4k(t+1)}\right)\right).$$

(c) The time snapshots are depicted in Figure 1. To give a better idea of how the solution looks like, a spacetime graph of the solution is shown in Figure 2.

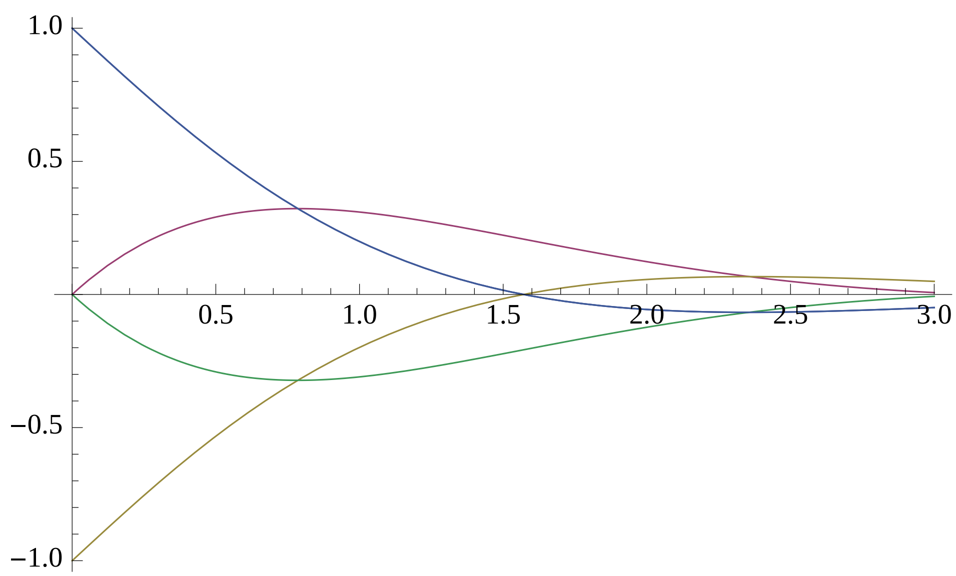

FIGURE 1. Time snapshots of the solution for 3.1:8c. Legend: $t = 0$ blue, $t = 1$ red, $t = 2$ yellow, $t = 3$ green, $t = 4$ blue again.

(d) The x-derivative of (4) is

$$\begin{aligned} u_x(x,t) &= -\sqrt{\tfrac{\omega}{2k}}\exp(-\sqrt{\tfrac{\omega}{2k}}x)\left(\cos(-\sqrt{\tfrac{\omega}{2k}}x + \omega t) - \sin(-\sqrt{\tfrac{\omega}{2k}}x + \omega t)\right) \\ &= -\sqrt{\tfrac{\omega}{k}}\exp(-\sqrt{\tfrac{\omega}{2k}}x)\cos(-\sqrt{\tfrac{\omega}{2k}}x + \omega t + \tfrac{\pi}{4}). \end{aligned}$$

Since $\exp(-\sqrt{\frac{\omega}{2k}}x) \neq$ for all x, the zeroes of $u_x(x,t)$ coincide with the zeroes of $\cos(-\sqrt{\frac{\omega}{2k}}x + \omega t + \frac{\pi}{4})$. The latter function is periodic in x with period $2\pi\sqrt{\frac{2k}{\omega}}$. This implies that the distance

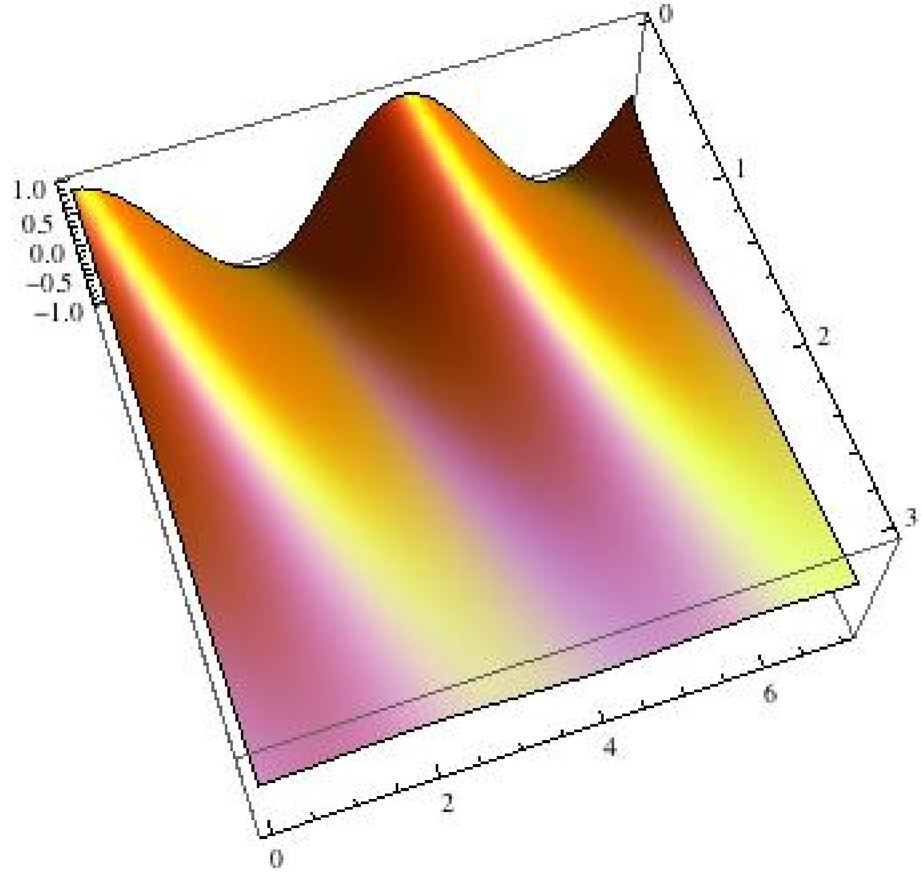

FIGURE 2. Spacetime graph of the solution for 3.1:8c. The t-axis is the one from left to right, the x-axis is from top to bottom, and the u-axis is directed towards the reader.

between consecutive local maxima is $2\pi\sqrt{\frac{2k}{\omega}}$ (there are two zeroes of u_x in one period, but one of the zeroes corresponds to a local minumum). As for the ratio of the values, we have

$$\frac{u(x_2,t)}{u(x_1,t)} = \frac{\exp(-\sqrt{\frac{\omega}{2k}}x_2)\cos(-\sqrt{\frac{\omega}{2k}}x_2+\omega t+\frac{\pi}{4})}{\exp(-\sqrt{\frac{\omega}{2k}}x_1)\cos(-\sqrt{\frac{\omega}{2k}}x_1+\omega t+\frac{\pi}{4})} = \exp(-\sqrt{\frac{\omega}{2k}}\cdot 2\pi\sqrt{\frac{2k}{\omega}}) = e^{-2\pi},$$

where we have taken into account the periodicity of cosine and the fact that $x_2-x_1 = 2\pi\sqrt{\frac{2k}{\omega}}$.

PROBLEM 3.2:1

Statement.

(a) Let $v(x,t)$ be any C^2 solution of $v_t = kv_{xx}$ ($0 \le x \le L$, $t \ge 0$), which satisfies the boundary conditions $v(0,t) = 0$ and $v(L,t) = 0$ (without initial condition). Show that for any t_1, t_2, with $t_2 \ge t_1 \ge 0$,

$$\int_0^L [v(x,t_2)]^2 \mathrm{d}x \le \int_0^L [v(x,t_1)]^2 \mathrm{d}x. \tag{5}$$

(b) Explain why the conclusion (5) still holds when the boundary conditions are replaced by any of the following pairs of boundary conditions:
   (i) $v_x(0,t) = v_x(L,t) = 0$,

(ii) $v_x(0,t) = v(L,t) = 0$,
(iii) $v_x(0,t) = h \cdot v(0,t)$ and $v(L,t) = 0$, where $h > 0$.

Solution. Let us define the function

$$E(t) = \int_0^L [v(x,t)]^2 \mathrm{d}x,$$

which can be called *energy*. Then (5) can be rephrased as

$$E(t_2) \le E(t_1), \qquad \text{for} \quad t_2 \ge t_1 \ge 0.$$

In other words, we have to show that E is a nondecreasing function of t. Let us calculate the time derivative of E as

$$\begin{aligned} E'(t) &= \int_0^L 2v(x,t)v_t(x,t)\mathrm{d}x = \int_0^L 2v(x,t)kv_{xx}(x,t)\mathrm{d}x \\ &= 2kv(L,t)v_x(L,t) - 2kv(0,t)v_x(0,t) - 2k\int_0^L |v_x(x,t)|^2\mathrm{d}x \\ &\le 2kv(L,t)v_x(L,t) - 2kv(0,t)v_x(0,t). \end{aligned} \tag{6}$$

We will show below that $E'(t) \le 0$ in various cases, which will then imply that E is nondecreasing. Note that each case requires a slightly different reasoning.

(a) We have

$$E'(t) \le 2k\underbrace{v(L,t)}_{=0}v_x(L,t) - 2k\underbrace{v(0,t)}_{=0}v_x(0,t) = 0.$$

(b)(i) Similarly, we have

$$E'(t) \le 2kv(L,t)\underbrace{v_x(L,t)}_{=0} - 2kv(0,t)\underbrace{v_x(0,t)}_{=0} = 0.$$

(b)(ii) We have

$$E'(t) \le 2k\underbrace{v(L,t)}_{=0}v_x(L,t) - 2kv(0,t)\underbrace{v_x(0,t)}_{=0} = 0.$$

(b)(iii) We have

$$E'(t) \le 2k\underbrace{v(L,t)}_{=0}v_x(L,t) - 2kv(0,t)\underbrace{v_x(0,t)}_{=hv(0,t)} = -2kh|v(0,t)|^2 \le 0.$$

Problem 3.2:2

Statement. State and prove a uniqueness theorem for the problem

$$u_t = u_{xx},$$

with the boundary conditions $u_x(0,t) = a(t)$ and $u_x(L,t) = b(t)$, and the initial condition $u(x,0) = f(x)$.

Solution. We will prove that any two C^2 solutions u_1 and u_2 must be equal to each other.

Supposing that u_1 and u_2 are two C^2 solutions of our problem, let us define $v = u_1 - u_2$. Then by subtracting the equations satisfied by u_2 from the corresponding ones for u_1, we see that v satisfies $v_t = v_{xx}$ with the boundary conditions $v_x(0,t) = v_x(L,t) = 0$, and the initial condition $v(x,0) = 0$. We want to show that v is zero everywhere. From Part (b)(i) of the previous problem, we have $E(t) \le E(0)$ for all $t \ge 0$, that is

$$E(t) = \int_0^L [v(x,t)]^2 \mathrm{d}x \le E(0) = \int_0^L [v(x,0)]^2 \mathrm{d}x = 0.$$

Since $v(x,t)$ is a continuous function of x, this implies that $v(x,t) = 0$ for all $0 \le x \le L$, and as $t \ge 0$ was arbitrary, we conclude that $v = 0$ everywhere.

Problem 3.2:3

Statement. Use maximum/minimum principles to deduce that the solution u of the problem

$$\begin{aligned} u_t &= k u_{xx}, \qquad (0 \le x \le \pi,\ t \ge 0), \\ u(0,t) &= u(\pi,t) = 0, \qquad (t \ge 0), \\ u(x,0) &= \sin x + \tfrac{1}{2}\sin 2x, \qquad (0 \le x \le \pi), \end{aligned}$$

satisfies $0 \le u(x,t) \le \frac{3}{4}\sqrt{3}$ for all $0 \le x \le \pi$ and $t \ge 0$.

Solution. We will show that $0 \le \sin x + \frac{1}{2}\sin 2x \le \frac{3}{4}\sqrt{3}$ for all $0 \le x \le \pi$, which would then imply by the maximum and minimum principles the desired bounds for the solution u. First of all, the representation

$$f(x) = \sin x + \tfrac{1}{2}\sin 2x = \sin x + \sin x \cos x = (1 + \cos x)\sin x,$$

reveals that $f(x) \ge 0$ for $0 \le x \le \pi$. Let us find the maximum of $f(x)$. We calculate

$$f'(x) = \cos x + \cos 2x = 2\cos^2 x + \cos x - 1,$$

whose zeros are given by $\cos x = \frac{-1 \pm 3}{4}$. This implies $x = \frac{2\pi}{3}$ and $x = \pi$. The point $x = \pi$ is clearly not a maximum because $f(\pi) = 0$. The other candidate gives

$$f(\tfrac{2\pi}{3}) = (1 + \cos\tfrac{2\pi}{3})\sin\tfrac{2\pi}{3} = \tfrac{3}{2} \cdot \tfrac{\sqrt{3}}{2}.$$

It is easy to see from the behaviour of the function $f'(x)$ or from an inspection of $f''(x)$ that $x = \frac{2\pi}{3}$ is the only maximum point in the interval $0 \le x \le \pi$.