

SOLUTIONS TO MIDTERM QUESTIONS

QUESTION 1

Statement. By using a method of your choice, solve the PDE

$$u_x \cos y + u_y = -u,$$

with the side condition $u(x, 0) = e^{-x^2}$. Sketch the characteristic curves. Interpreting the y variable as time, describe in words how the initial “bump” function e^{-x^2} evolves as time runs.

Solution. If we want to describe any characteristic curve by a function $y = y(x)$, then we would have to solve $y'(x) = 1/\cos y$. One can of course proceed and solve the problem in this way, but one has to be extra careful because of the zeroes of $\cos y$. Let us choose a bit different approach, and describe the characteristic curves by $x = x(y)$. Then the characteristic equation is

$$\frac{dx}{dy} = \cos y,$$

which can easily be solved as

$$x(y) = \sin y + C = \sin y + x(0),$$

where we have identified the constant C as the value of $x(y)$ at $y = 0$.

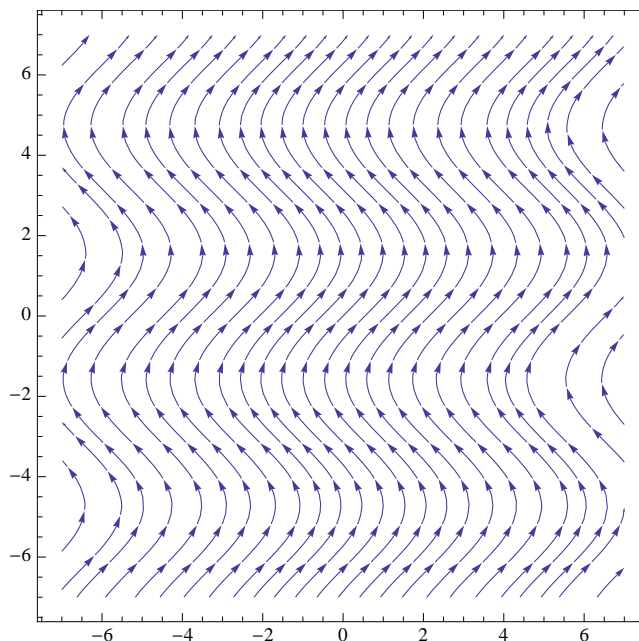


FIGURE 1. Characteristic curves for Question 1. The x -axis runs horizontally, and the y -axis vertically.

Supposing that u is a solution of the PDE, and by differentiating u along a characteristic, we get

$$\frac{d}{dy}u(x(y), y) = u_x(x(y), y)\frac{dx(y)}{dy} + u_y(x(y), y) = -u(x(y), y),$$

and so

$$u(x(y), y) = Be^{-y}.$$

Plugging in the side condition gives

$$u(x(0), 0) = B = e^{-x(0)^2}.$$

Now suppose that (x, y) is an arbitrary point in \mathbb{R}^2 . Then the characteristic curve passing through this point must have $x(0) = x - \sin y$. From this, we infer the solution

$$u(x, y) = e^{-(x-\sin y)^2} e^{-y}.$$

As time evolves, “bump” would decay (or “flatten”) exponentially. At the same time, the peak of the “bump” would move back and forth in x -space by the law $x = \sin y$.

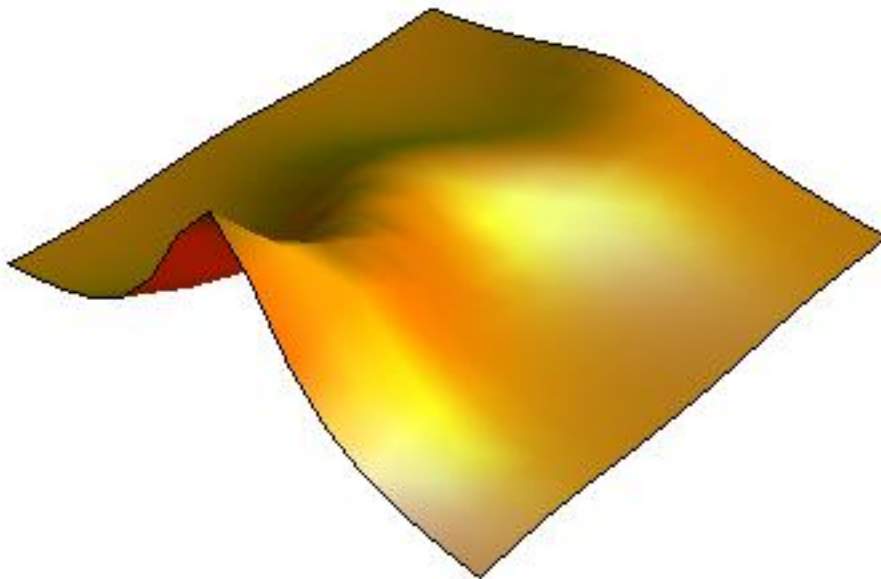


FIGURE 2. Depiction of the solution for Question 1. The x -axis is the one directed from top left to bottom right, and the y -axis is from bottom left to top right. The value of $u(x, y)$ is the elevation, which is a bit exaggerated for visualization purposes. In true scale $u(x, t)$ decays in time so fast that the wiggly movement of the peak would not be noticeable.

QUESTION 2

Statement. Write down a solution of the heat equation

$$u_t = u_{xx},$$

for $0 \leq x \leq \pi$ and $t > 0$, satisfying the homogeneous Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad (t > 0),$$

and the initial condition

$$u(x, 0) = \sum_{n=1}^N \frac{1 + (-1)^n}{n^s} \sin \frac{nx}{2}, \quad (0 \leq x \leq \pi),$$

where N is a positive integer, and s is a real number, both considered to be given. *Hint:* The usual formula with product solutions would not directly apply, because $\frac{nx}{2}$ is not of the form mx with integer m .

Solution. The key observation here is that $1 + (-1)^n = 0$ for odd n . This leaves only the even terms, hence the initial condition is

$$u(x, 0) = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2}{(2m)^s} \sin \frac{2mx}{2} = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2}{(2m)^s} \sin mx, \quad (0 \leq x \leq \pi),$$

where $\lfloor N/2 \rfloor$ is the largest integer not exceeding $N/2$. Now there is no problem with applying the product solution formula, and we get

$$u(x, t) = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2e^{-mt}}{(2m)^s} \sin mx, \quad (0 \leq x \leq \pi, t \geq 0).$$

QUESTION 3

Statement. Consider the PDE

$$u_t = \kappa u_{xx} + \alpha u + f,$$

on the spatial interval $0 < x < L$ (with, say $t > 0$), where κ , α , and L are positive constants, and f is a given function. By a change of variables, transform the problem into an equivalent problem

$$v_t = \varepsilon v_{xx} + v + g,$$

on the spatial interval $0 < x < 1$. Give formulas relating the new quantities v , ε , and g to the old ones.

Solution. Let us put

$$v(x, t) = u(Lx, \lambda t),$$

where λ is a real number that will be determined. Note that the scaling factor L for the x variable is immediately obvious, as we want x in $v(x, t)$ to vary between $0 < x < 1$. We have

$$v_t(x, t) = \lambda u_t(Lx, \lambda t), \quad v_{xx}(x, t) = L^2 u_{xx}(Lx, \lambda t),$$

and so

$$\begin{aligned} v_t(x, t) &= \lambda u_t(Lx, \lambda t) = \kappa \lambda u_{xx}(Lx, \lambda t) + \alpha \lambda u(Lx, \lambda t) + \lambda f(x, t) \\ &= \kappa \lambda L^{-2} v_{xx}(x, t) + \alpha \lambda v(x, t) + \lambda f(x, t). \end{aligned}$$

In order to have the coefficient in front of v equal to 1, we need $\lambda = \alpha^{-1}$. This fixes the rest of the coefficients, and we conclude that the transformations

$$v(x, t) = u\left(Lx, \frac{t}{\alpha}\right), \quad g(x, t) = \frac{f(x, t)}{\alpha}, \quad \varepsilon = \frac{\kappa}{\alpha L^2},$$

satisfy the required conditions.