SOLUTIONS TO PROBLEMS FROM ASSIGNMENT 2

Problems 1.3:2d and 1.3:3d

Statement. Find general solutions of $yu_{xy} + 2u_x = x$ using ODE techniques, as well as its particular solution satisfying the side conditions $u(x,1) = 0$ and $u(0,y) = 0$.

Solution. Integrating the equation with respect to x gives

$$yu_y + 2u = \frac{x^2}{2} + C(y),$$

where $C(y)$ is an arbitrary function of y. Now we use the integrating factor $m(y) = y$ to proceed

$$(y^2u)_y = y^2u_y + 2yu = y(yu_y + 2u) = \frac{1}{2}x^2y + yC(y).$$

Hence we have

$$y^2u(x,y) = \frac{1}{4}x^2y^2 + \int yC(y)\mathrm{d}y + B(x),$$

and the solution is

$$u(x,y) = \frac{1}{4}x^2 + A(y) + y^{-2}B(x),$$

where A and B are differentiable functions. This is indeed a solution, since it gives

$$u_x(x,y) = \frac{1}{2}x + y^{-2}B'(x), \qquad \text{and} \qquad u_{xy}(x,y) = -2y^{-3}B'(x).$$

Now let us plug in the side conditions. We get

$$u(x,1) = \frac{1}{4}x^2 + A(1) + B(x) = 0, \qquad \Rightarrow \qquad B(x) = -A(1) - \frac{1}{4}x^2,$$

and

$$u(0,y) = A(y) + y^{-2}B(0) = 0, \qquad \Rightarrow \qquad A(y) = -y^{-2}B(0) = y^{-2}A(1),$$

leading to the final solution

$$u(x,y) = \frac{1}{4}x^2 + y^{-2}A(1) + y^{-2}\left(-A(1) - \frac{1}{4}x^2\right) = \frac{x^2}{4}\left(1 - \frac{1}{y^2}\right).$$

Problem 2.1:2b

Statement. Find the particular solution of $u_x + 2u_y - 4u = e^{x+y}$ satisfying $u(0,y) = y^2$.

Date: Winter 2012.

Solution. To find the general solution, let us use the coordinate transformation

$$x = \xi, \qquad y = \eta + 2\xi.$$

The variable y transforms into the new variable η, and x stays the same. The useful property of this transformation is

$$u_\xi = u_x x_\xi + u_y y_\xi = u_x + 2u_y,$$

which converts the equation into

$$u_\xi - 4u = e^{3\xi+\eta}.$$

This can be solved by using the integrating factor $m(\xi) = e^{-4\xi}$ as

$$(e^{-4\xi}u)_\xi = e^{-4\xi}(u_\xi - 4u) = e^{\eta-\xi},$$

leading to

$$e^{-4\xi}u(\xi,\eta) = -e^{\eta-\xi} + C(\eta), \qquad \Rightarrow \qquad u(\xi,\eta) = -e^{3\xi+\eta} + e^{4\xi}C(\eta).$$

Taking into account that $\xi = x$ and $\eta = y - 2x$, we have

$$u(x,y) = -e^{x+y} + e^{4x}C(y-2x).$$

Now plugging the side condition into this, we get

$$u(0,y) = -e^y + C(y) = y^2, \qquad \Rightarrow \qquad C(y) = y^2 + e^y.$$

So the final solution is

$$u(x,y) = -e^{x+y} + e^{4x}\left((y-2x)^2 + e^{y-2x}\right).$$

Problem 2.1:3

Statement. Show that the PDE $u_x + u_y - u = 0$ with the side condition $u(x,x) = \tan(x)$ has no solution.

Solution. An intuitive explanation of the situation is that the side condition is given on a characteristic line, and is inconsistent with the differential equation. Let us define the function $f(t) = u(x(t), y(t))$, where $x(t) = t$ and $y(t) = t$. We calculate the derivative of this function from the PDE as

$$\begin{aligned} f'(t) &= u_x(x(t),y(t))x'(t) + u_y(x(t),y(t))y'(t) \\ &= u_x(x(t),y(t)) + u_y(x(t),y(t)) \\ &= u(x(t),y(t)) \\ &= f(t), \end{aligned}$$

which means that $f(t) = f(0)e^t$. From the side condition, we have $f(t) = \tan(t)$, which clearly does not satisfy $f' = f$. Hence, there is no function f that satisfies the both conditions. As f is simply the value of $u(x,y)$ on the "diagonal" $\{x = y\}$, this makes the existence of u impossible.

Problem 2.1:8

Statement. (a) Show that the PDE $u_x = 0$ has no solution which is C^1 everywhere and satisfies the side condition $u(x,x^2) = x$.

(b) Find a solution of the problem in (a) which is valid in the first quadrant $x > 0$, $y > 0$.

(c) Explain the results of (a) and (b) in terms of the intersections of the side condition curve and the characteristic lines.

Solution. (a) The PDE simply means that u does not depend on x, hence the general solution is $u(x, y) = g(y)$ for any function g. Similarly to the previous problem, let us define the function $f(t) = u(x(t), y(t))$ with $x(t) = t$ and $y(t) = t^2$. From the PDE, we have $f(t) = g(t^2) = g((-t)^2) = f(-t)$, meaning that f is an even function. However, the side condition gives $f(t) = t$, which is a nontrivial odd function. Hence, there is no function f that satisfies the both conditions.

(b) Enforcing the side condition to the general solution $u(x, y) = g(y)$ gives $u(x, x^2) = g(x^2) = x$, and since $x > 0$ we get $g(y) = \sqrt{y}$ for $y > 0$. So the solution is $u(x, y) = \sqrt{y}$ for $x > 0$ and $y > 0$.

(c) The characteristic lines are simply the horizontal lines $\{y = \text{const}\}$, and the PDE requires the solutions to be constant along those lines. The side condition curve is the parabola $y = x^2$, and it intersects each characteristic line in the upper half plane $\{y > 0\}$ twice. This means that in order for a solution to exist, the side condition must be symmetric with respect to the y-axis. However, the side condition $u(x, x^2) = x$ does not have the required symmetry.

Problem 2.1:9

Statement. (a) Show that the PDE $u_x = 0$ has no solution which is C^1 everywhere and satisfies the side condition $u(x, x^3) = x$, even though the side condition curve $y = x^3$ intersects each characteristic lines.

(b) Part (a) demonstrates the necessity of the transversality condition on the intersections of the side condition curve with the characteristic lines. Explain why.

Solution. (a) Let us define the function $f(t) = u(x(t), y(t))$ with $x(t) = t$ and $y(t) = t^3$. From the PDE, the derivative of this function is

$$\begin{aligned} f'(t) &= u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) \\ &= u_x(x(t), y(t)) + 3t^2 u_y(x(t), y(t)) \\ &= 3t^2 u_y(x(t), y(t)), \end{aligned}$$

so in particular, $f'(0) = 0$. The side condition gives $f(t) = t$, implying that $f'(t) = 1$ for all t. Hence, there is no function f that satisfies the both conditions. Note that the same reasoning would have worked also for the previous problem.

(b) Consider a situation where a characteristic curve intersects the side condition curve twice, and imagine that the two intersection points are getting closer and closer together, as a result of varying the side condition curve in some way. In the limit it would produce a situation where a characteristic curve intersects the side condition curve tangentially. Since in the case with two intersections the solution u must have the same value at the two points, intuitively speaking, in the limit case the tangential derivative of u at the intersection point must vanish. So if the tangential derivative of the side condition at the intersection point is nonzero, a contradiction would arise.

Problem 2.1:10

Statement. (a) Show that a solution of the homogeneous PDE $au_x + bu_y + cu = 0$ cannot be zero at one, and only one, point on the plane.

(b) If $c = 0$ in the PDE in (a), then show that the graph $z = u(x, y)$ of a solution u (defined everywhere) is a surface composed of horizontal parallel lines.

Solution. We will assume that at least one of a and b is nonzero, i.e., $a^2+b^2 > 0$. If $a = b = 0$, then the equation reduces to $cu = 0$, which implies $u \equiv 0$ provided $c \neq 0$. There is no point in considering the case $a = b = c = 0$.

(a) Suppose that u is a solution of the PDE, and suppose that $u(x_0, y_0) = 0$ at some point (x_0, y_0). We are done if we can produce a point (x, y) such that $u(x, y) = 0$ and $(x, y) \neq (x_0, y_0)$. Let us define the function $f(t) = u(x(t), y(t))$, with $x(t) = x_0 + at$ and $y(t) = y_0 + bt$. It is obvious that the point $(x(t), y(t))$ is different from (x_0, y_0) unless $t = 0$, and coincides with (x_0, y_0) if $t = 0$. In particular, we have $f(0) = 0$. Let us calculate the derivative of f, by using the PDE as

$$\begin{aligned} f'(t) &= u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) \\ &= au_x(x(t), y(t)) + bu_y(x(t), y(t)) \\ &= -cu(x(t), y(t)) \\ &= -cf(t). \end{aligned}$$

This implies that $f(t) = f(0)e^{-ct}$, but as $f(0) = 0$, we get $f(t) = 0$ for all t. Thus for example, choosing $t = 1$ gives $f(1) = u(x(1), y(1)) = u(x_0 + a, y_0 + b) = 0$.

(b) Let (x_0, y_0) be an arbitrary point on the plane, and define the function f as in (a). Then we have $f'(t) = -cf(t)$, and since $c = 0$, we get $f(t) = \text{const}$ for all t. Note that the derivation of the equation $f' = -cf$ in (a) does not depend on the assumption $u(x_0, y_0) = 0$, and the latter assumption was only used to infer $f(0) = 0$. Now $f(t) = \text{const}$ means in terms of u that $u(x_0 + at, y_0 + bt) = u(x_0, y_0)$ for all t. The graph Γ of u is the surface in $\mathbb{R}^3$ defined by

$$\Gamma = \{(x, y, u(x, y)) : (x, y) \in \mathbb{R}^2\}.$$

The above consideration says that if the point (x_0, y_0, z_0) is in the graph, that is to say if $z_0 = u(x_0, y_0)$, then the line $\ell(x_0, y_0) = \{(x_0 + at, y_0 + bt, z_0) : t \in \mathbb{R}\}$ is entirely contained in the graph. We see that the graph Γ is simply the union of all the lines $\ell(x_0, y_0)$ as the point (x_0, y_0) runs over the plane $\mathbb{R}^2$.