SOLUTIONS TO PROBLEMS 8B AND 9 FROM ASSIGNMENT 1

Problem 8b

Statement. If $a(x)$, $b(x)$ and $c(x)$ are continuous with $a(x)$ never zero, then the ODE $a(x)y'' + b(x)y' + c(x)y = 0$ has a unique solution $y(x)$ with given values for $y(x_0)$ and $y'(x_0)$. Assuming this, show that no solution of this ODE can have a graph which is tangent to the x-axis at some point, unless the solution is identically zero.

Solution. Suppose that the graph of $y(x)$ is tangent to the x-axis at x_0, meaning that $y(x_0) = 0$ and $y'(x_0) = 0$. Then we observe that the identically-zero function $y(x) \equiv 0$ is a solution of the ODE with $y(x_0) = 0$ and $y'(x_0) = 0$. By uniqueness, this is the only solution of our ODE. In other words, there is no function $y(x)$ that is not identically zero and satisfies the ODE together with the conditions $y(x_0) = 0$ and $y'(x_0) = 0$.

Problem 9

Statement. (a) If $ar^2 + br + c = 0$ has only one root (of multiplicity 2) $r = -\frac{b}{2a}$, show that $f(x)e^{rx}$ is a solution of $ay'' + by' + cy = 0$ if and only if $f''(x) = 0$.

(b) For distinct numbers r_1 and r_2 observe that

$$\lim_{r_2 \to r_1} \frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1} = xe^{r_1 x}.$$

How is this observation related to the result in part (a) ?

Solution. (a) Let $y(x) = f(x)e^{rx}$. We calculate

$$y'(x) = (f'(x) + f(x)r)e^{rx}, \qquad y''(x) = (f''(x) + f'(x)r)e^{rx} + (f'(x) + f(x)r)re^{rx},$$

and

$$\begin{aligned} ay'' + by' + cy &= a(f'' + rf')e^{rx} + ar(f' + rf)e^{rx} + b(f' + rf)e^{rx} + cfe^{rx} \\ &= af''e^{rx} + (2ar + b)f'e^{rx} + (ar^2 + br + c)fe^{rx} \\ &= af''e^{rx}, \end{aligned}$$

since $2ar + b = 0$ and $ar^2 + br + c = 0$. Here we omitted the variable x from $f(x)$ etc. for the sake of readability. From the preceding calculation, we infer that since ae^{rx} is never zero, the quantity $ay'' + by' + cy$ vanishes exactly where f'' vanishes.

To conclude, supposing that A and B are constants, the function

$$y(x) = Ae^{rx} + Bxe^{rx}, \tag{1}$$

is a solution of the ODE $ay'' + by' + cy = 0$. Moreover, any solution of this (2nd order) ODE has the form (1), since the functions e^{rx} and xe^{rx} are linearly independent.

(b) One possible way to compute the limit is as follows. We write

$$e^{r_2 x} - e^{r_1 x} = e^{r_1 x}(e^{(r_2 - r_1)x} - 1),$$

and with $\alpha = r_2 - r_1$, look at the Taylor expansion

$$e^{\alpha x} = 1 + \alpha x + \frac{1}{2}\alpha^2 x^2 + \ldots = 1 + \alpha x + \alpha g(\alpha),$$

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where we consider x as fixed and α small (as we want to send α to 0), and $g(\alpha)$ is a continuous function of α that goes to 0 as $\alpha \to 0$. Thus we have

$$\frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1} = \frac{e^{\alpha x} - 1}{\alpha} e^{r_1 x} = \frac{\alpha x + \alpha g(\alpha)}{\alpha} e^{r_1 x} = (x + g(\alpha)) e^{r_1 x},$$

which makes it obvious that the limit as $\alpha \to 0$ is $x e^{r_1 x}$.

As for how this is related to part (a), if the roots r_1 and r_2 of $ar^2 + br + c = 0$ are distinct, the general solution of $ay'' + by' + cy = 0$ is given by

$$y(x) = Ce^{r_1 x} + De^{r_2 x},$$

with C and D constants. With this result as our starting point, we can try to extract some information on the case $r_2 = r_1$ by taking the limit $\alpha = r_2 - r_1 \to 0$. Having in mind the limit we just computed, let us put $D = -C = 1/(r_2 - r_1)$ to get

$$y(x) = Ce^{r_1 x} + De^{r_2 x} = \frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1} \to x e^{r_1 x}.$$

We have produced a new linearly independent solution by suitably scaling the constants C and D as $r_2 \to r_1$.