SOLUTIONS TO SELECTED PROBLEMS FROM ASSIGNMENTS 3, 4

Problem 5 from Assignment 3

Statement. Let Ω be an n-dimensional bounded domain with smooth boundary. Show that the eigenvalues of the Laplacian on Ω with homogeneous Neumann boundary condition cannot be positive. Show also that the eigenvalues of the Laplacian on Ω with homogeneous Dirichlet boundary condition are strictly negative. (*Hint* for the Dirichlet case: The Laplace equation has a unique solution with the homogeneous Dirichlet boundary condition.)

Solution. Let us denote by $\nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal vector to the boundary $\partial\Omega$. Note that ν is a vector field that is defined at each point of the boundary $\partial\Omega$. That u is an *eigenfunction* of the Laplacian with the homogeneous Neumann condition means that there is a number $\lambda \in \mathbb{R}$ called the *eigenvalue* corresponding to u, such that

$$\Delta u = \lambda u \quad \text{in} \quad \Omega, \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $\partial_\nu u$ is the normal derivative

$$\partial_\nu u = \nu_1 u_{x_1} + \ldots + \nu_n u_{x_n}.$$

We require u to be not identically zero, since the zero function $u \equiv 0$ satisfies (1) for any $\lambda \in \mathbb{R}$. Recall *Green's first identity*

$$\int_\Omega v\Delta u = \int_{\partial\Omega} v\partial_\nu u - \int_\Omega \nabla v \cdot \nabla u,$$

which can be proven by applying the divergence theorem to the vector field $v\nabla u$. Let us multiply the eigenvalue equation $\Delta u = \lambda u$ by u, and integrate over the domain Ω, to get

$$\int_\Omega u\Delta u = \lambda \int_\Omega u^2.$$

Applying Green's identity to the left hand side gives

$$\int_{\partial\Omega} u\partial_\nu u - \int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2,$$

which then implies

$$\lambda \int_\Omega u^2 = -\int_\Omega |\nabla u|^2, \tag{2}$$

because the boundary integral term is zero due to the homogeneous Neumann condition. From the last equality we infer $\lambda \leq 0$ since u must be nonzero in order to be an eigenfunction.

For the Dirichlet case, by following a similar reasoning we can conclude that $\lambda \leq 0$. If there is an eigenfunction u with $\lambda = 0$, it must satisfy

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Clearly, the identically-zero function $u \equiv 0$ satisfies this equation, and as given in Hint, it is the only solution. So there is no nontrivial function u that satisfies $\Delta u = 0$, showing that all eigenvalues are nonzero (therefore $\lambda < 0$).

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A variation of Problem 4 from Assignment 4

Statement. Solve the Dirichlet boundary value problem for the Laplace equation $\Delta u = 0$ in the region between two concentric spheres of radii 1 and 2. What simplifications do we get in the solution process if the Dirichlet boundary data are independent of the azimuthal angle (longitude) φ?

Solution. Employing spherical coordinates, the equation we need to solve is

$$\Delta u = 0, \quad \text{in} \quad \{(r,\theta,\varphi) : 1 < r < 2\}, \quad \text{and} \quad u(1,\theta,\varphi) = f(\theta,\varphi), \quad u(2,\theta,\varphi) = g(\theta,\varphi).$$

Recall that the 3-dimensional Laplacian in spherical coordinates is

$$\begin{aligned}\Delta u &= \frac{(r^2 u_r)_r}{r^2} + \frac{(u_\theta \sin\theta)_\theta}{r^2\sin\theta} + \frac{u_{\varphi\varphi}}{r^2\sin^2\theta} \\ &= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}.\end{aligned}$$

Let us look for a solution that can be written as $u(r,\theta,\varphi) = R(r)M(\theta,\varphi)$. Substituting this into the equation $\Delta u = 0$, we get

$$R''M + \frac{2}{r}R'M + \frac{1}{r^2}RM_{\theta\theta} + \frac{\cot\theta}{r^2}RM_\theta + \frac{1}{r^2\sin^2\theta}RM_{\varphi\varphi} = 0,$$

and multiplying by $\frac{r^2}{RM}$ and rearranging, we have

$$-\frac{r^2R''}{R} - \frac{2rR'}{R} = \frac{M_{\theta\theta}}{M} + \frac{M_\theta\cot\theta}{M} + \frac{M_{\varphi\varphi}}{M\sin^2\theta} = \lambda.$$

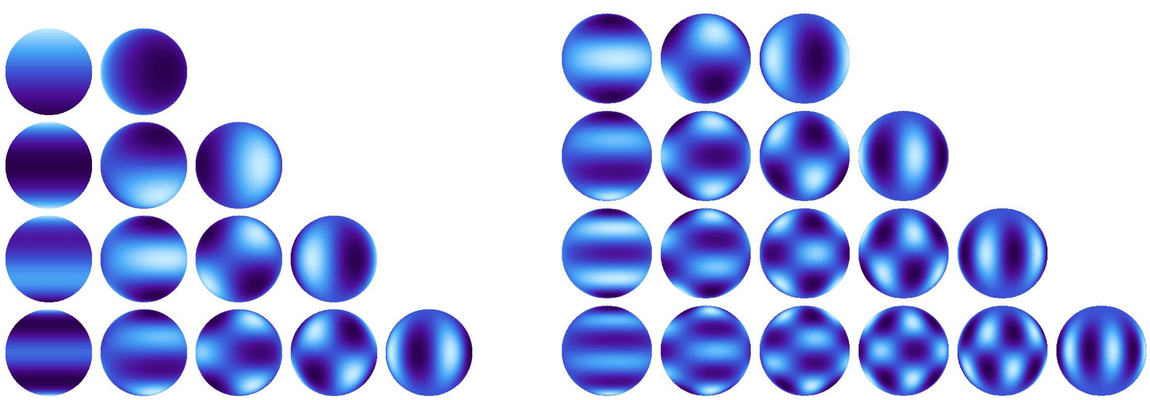
We recognise the equation given by the second equality sign as the equation for the *spherical harmonics*, so for each $n = 0,1,\ldots$, and for each $m = 0,\pm1,\ldots,\pm n$, we have the solution

$$M(\theta,\varphi) = Y_{n,m}(\theta,\varphi), \qquad \text{with} \qquad \lambda = -n(n+1).$$

Recall that the spherical harmonics are explicitly given by

$$Y_{n,m}(\theta,\varphi) = \sin^{|m|}(\theta)\, P_n^{(|m|)}(\cos\theta)\,\Phi_m(\varphi), \qquad \text{where} \qquad \Phi_m(\varphi) = \begin{cases}\cos(m\phi) & \text{for } m \geq 0,\\ \sin(m\phi) & \text{for } m < 0,\end{cases}$$

and where P_n is the *Legendre polynomial* of degree n.



(a) $n = 1,2,3,4$ from top to bottom, and $m = 0,\ldots,n$ from left to right

(b) $n = 3,4,5,6$ from top to bottom, and $m = 1,\ldots,n$ from left to right

Figure 1. Spherical harmonics Y_{nm}

Taking into account that $\lambda = -n(n+1)$, the equation for R now reads

$$r^2 R'' + 2rR' = n(n+1)R.$$

Trying the form $R(r) = r^\alpha$ gives

$$\alpha(\alpha-1) + 2\alpha = n(n+1),$$

which has the solutions $\alpha = n$ and $\alpha = -(n+1)$. We conclude that for each $n \in \mathbb{N}_0$, and for each $m \in \mathbb{Z} \cap [-n, n]$, we have two independent solutions

$$u(r,\theta,\varphi) = r^n Y_{n,m}(\theta,\varphi), \qquad \text{and} \qquad u(r,\theta,\varphi) = r^{-(n+1)} Y_{n,m}(\theta,\varphi),$$

and so the general solution is given by a linear combination of all of those:

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(A_{n,m} r^n + \frac{B_{n,m}}{r^{n+1}} \right) Y_{n,m}(\theta,\varphi).$$

The coefficients $A_{n,m}$ and $B_{n,m}$ are to be found from the boundary conditions. This means, assuming the following expansions

$$f(\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{f}_{n,m} Y_{n,m}(\theta,\varphi), \qquad g(\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \hat{g}_{n,m} Y_{n,m}(\theta,\varphi),$$

we have

$$A_{n,m} + B_{n,m} = \hat{f}_{m,n}, \qquad \text{and} \qquad A_{n,m} 2^n + \frac{B_{n,m}}{2^{n+1}} = \hat{g}_{m,n}.$$

This is easily solved as

$$A_{n,m} = \frac{2^{n+1}\hat{g}_{m,n} - \hat{f}_{m,n}}{2^{2n+1} - 1}, \qquad \text{and} \qquad B_{n,m} = \frac{2^{2n+1}\hat{f}_{m,n} - 2^{n+1}\hat{g}_{m,n}}{2^{2n+1} - 1}.$$

Now, for the **second part of the question**, when the boundary data are independent of φ, the solution must also be independent of φ. Then one way to solve the problem would be to first solve the problem in full generaility as in the first part of the question, and then impose the φ-independence on the resulting solution. The spherical harmonics $Y_{n,m}$ that do not depend on φ are the ones with $m = 0$, i.e, the *zonal harmonics*

$$Y_{n,0}(\theta,\varphi) = Z_n(\theta) \equiv P_n(\cos\theta),$$

hence the solution is simply

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta).$$

An alternaitve, more direct approach is to impose the φ-independence right from the start; so we look for a solution $u(r,\theta) = R(r)\Theta(\theta)$ rather than $u(r,\theta,\varphi) = R(r)M(\theta,\varphi)$, and we get

$$-\frac{r^2 R''}{R} - \frac{2rR'}{R} = \frac{\Theta''}{\Theta} + \frac{\Theta' \cot\theta}{\Theta} = \lambda.$$

We recognise the equation for Θ as the equation leading to the *Legendre equation*

$$(1-x^2)y'' - 2xy' = \lambda y,$$

where $y(x) = \Theta(\theta)$ with $x = \cos\theta$, meaning that we have a solution $\Theta(\theta) = P_n(\cos\theta)$ with $\lambda = -n(n+1)$ for each $n = 0, 1, \ldots$. The rest of the solution proceeds completely in parallel to the solution of the first part of the question.

Problem 5 from Assignment 4

Statement. Consider a bounded domain, and assume that the homogeneous Neumann boundary condition is imposed on all problems we consider in this question. Show that the Laplacian posseses an eigenvalue equal to zero, i.e., show that there is a nonzero function v (having the homogeneous Neumann boundary condition) such that $\Delta v = 0$. What would be the effect of this zero eigenvalue on the solutions of the heat equation $u_t = \Delta u$, and of the wave equation $u_{tt} = \Delta u$? What can you say about the solvability of the Poisson equation $\Delta u = f$?

Solution. Let us denote the domain by $\Omega \subset \mathbb{R}^n$, and let $\nu = (\nu_1, \ldots, \nu_n)$ be the outward unit normal vector to the boundary $\partial\Omega$. The Laplace eigenproblem under the Neumann condition reads

$$\Delta v = \lambda v, \quad \text{in} \quad \Omega, \quad \text{and} \quad \partial_\nu v = 0, \quad \text{on} \quad \partial\Omega, \tag{3}$$

where $\partial_\nu v$ is the normal derivative

$$\partial_\nu v = \nu_1 v_{x_1} + \ldots + \nu_n v_{x_n}.$$

To show that the Laplacian posseses a zero eigenvalue, we need to find a nontrivial function v that satisfies (3) with $\lambda = 0$. By working out the one dimensional case $\Delta v \equiv v'' = 0$ and $v'(0) = v'(1) = 0$ on the interval $(0, 1)$, whose solutions are the constant functions, we guess that in general case the constant functions might be eigenfunctions with $\lambda = 0$. It is straightforward to see that for $v = \text{const}$ we have

$$\Delta v = v_{x_1x_1} + \ldots + v_{x_nx_n} = 0, \qquad \partial_\nu v = \nu_1 v_{x_1} + \ldots + \nu_n v_{x_n} = 0,$$

showing that the Laplacian has a zero eigenvalue, with constants being one of the correspong-ing eigenfunctions, if there is any other than constants. Note that we count all constants together as one eigenfunction, since if v satisfies (3), then so does kv for any $k \in \mathbb{R}$. To check if there is any eigenfunction other than constants, recall the identity (2) we have derived for the Laplace eigenfunctions with the Neumann boundary condition, which becomes

$$\int_\Omega |\nabla v|^2 = 0,$$

in the current case with $\lambda = 0$. The quantity under integration sign is nonnegative, meaning that the quantity must be zero everywhere. In other words, we have the gradient of v vanishing everywhere, and so v must be constant. This argument can also be used from the start to discover that constants are eigenfunctions with zero eigenvalue.

Let $v_0, v_1, \ldots$ be the Laplace eigenfunctions with the corresponding eigenvalues $\lambda_0, \lambda_1, \ldots$, i.e., for $k \in \mathbb{N}_0$, let

$$\Delta v_k = \lambda_k v_k \quad \text{in} \quad \Omega, \quad \text{and} \quad \partial_\nu v_k = 0 \quad \text{on} \quad \partial\Omega.$$

Suppose, for concreteness that the eigenvalues are arranged as follows

$$\lambda_0 = 0 > \lambda_1 \geq \lambda_2 \geq \ldots.$$

In particular, this means that v_0 is a constant function, so we can assume that $v_0 = 1$. We know that $\lambda_k \to -\infty$ as $k \to \infty$, and that any function $f : \Omega \to \mathbb{R}$ with finite energy can be decomposed in terms of the eigenfunctions as

$$f = \sum_{k=0}^{\infty} \hat{f}_k v_k, \tag{4}$$

where we call $\{\hat{f}_k\}$ the *coordinates* of f in the basis $\{v_k\}$, and we refer to $\hat{f}_k v_k$ as the *k-th mode* of f. Accordingly, f is the sum of its individual modes, and one can think of $|\hat{f}_k|$ as measuring the size of the k-th mode. It is a general fact that as k gets large the function v_k

becomes more and more oscillatory, and so the modes with large (or small) k are called high (or low) frequency modes.

Let us first treat the *Poisson problem*

$$\Delta u = f \quad \text{in} \quad \Omega, \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega.$$

Expanding the unknown solution u in terms of the eigenfunctions, we get

$$u = \sum_{k=0}^{\infty} \hat{u}_k v_k \qquad \Rightarrow \qquad \Delta u = \sum_{k=0}^{\infty} \hat{u}_k \Delta v_k = \sum_{k=0}^{\infty} \hat{u}_k \lambda_k v_k = \sum_{k=0}^{\infty} \hat{f}_k v_k.$$

Equating the individual coordinates, we have

$$\lambda_k \hat{u}_k = \hat{f}_k \qquad \Rightarrow \qquad 0 \cdot \hat{u}_0 = \hat{f}_0 \quad \text{and} \quad \hat{u}_k = \frac{\hat{f}_k}{\lambda_k} \quad \text{for} \quad k = 1, 2, \ldots,$$

where we have taken into account that $\lambda_0 = 0$ and $\lambda_k \neq 0$ for $k > 0$. We see that as a result of division by λ_k, the high frequency modes of u are much smaller than the corresponding modes of f, i.e., the coordinates of u decays faster than the coordinates of f, with the difference in the decay rates given by the growth rate of $|\lambda_k|$. More crucially, we also see that because of the existence of the zero eigenvalue, in order for the Poisson equation to be solvable, the right hand side f must have a vanishing zero-mode, i.e., we must require $\hat{f}_0 = 0$. If this requirement is met, the function

$$u = \hat{u}_0 v_0 + \sum_{k=1}^{\infty} \frac{\hat{f}_k}{\lambda_k} v_k = \hat{u}_0 \cdot 1 + \sum_{k=1}^{\infty} \frac{\hat{f}_k}{\lambda_k} v_k,$$

with any $\hat{u}_0 \in \mathbb{R}$ satisfies the Poisson equation. If u is a solution, then for any constant k, $u + k$ is also a solution. Finally, let us clarify what $\hat{f}_0 = 0$ means. By multiplying (4) by v_0 and integrating over Ω, and then taking into account the orthogonality of the eigenfunctions, we get

$$\hat{f}_0 \int_\Omega |v_0|^2 = \int_\Omega f v_0 \qquad \Rightarrow \qquad \hat{f}_0 = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f,$$

that is, $\hat{f}_0$ is just the average of f over the domain Ω. Here $\mathrm{vol}(\Omega)$ denotes the volume of Ω. We conclude that the necessary and sufficient condition for solvability of the Poisson problem with the homogeneus Neumann boundary condition is that the right hand side function has average zero.

Now let us turn to the *heat equation*

$$u_t = \Delta u, \qquad u|_{t=0} = f,$$

with the homogeneus Neumann boundary condition. Since the solution u depends both on the spatial variable $x \in \Omega$ and time t, the coordinates (and so the modes) of u must depend on t. More specifically, we have

$$u(x,t) = \sum_{k=0}^{\infty} \hat{u}_k(t) v_k(x) \qquad \Rightarrow \qquad u_t = \sum_{k=0}^{\infty} \hat{u}_k' v_k, \qquad \Delta u = \sum_{k=0}^{\infty} \lambda_k \hat{u}_k v_k,$$

where $\hat{u}_k'$ denotes the time derivative of $\hat{u}_k$. Substituting the latter formulas into the heat equation, we infer

$$\hat{u}_k' = \lambda_k \hat{u}_k,$$

which is solved by $\hat{u}_k(t) = A_k e^{\lambda_k t}$. From the initial condition we get $A_k = \hat{f}_k$, and so

$$u(x,t) = \sum_{k=0}^{\infty} \hat{f}_k e^{\lambda_k t} v_k(x) = \hat{f}_0 + \sum_{k=1}^{\infty} \hat{f}_k e^{\lambda_k t} v_k(x),$$

where we have taken into account that $\lambda_0 = 0$ and $v_0 = 1$. The effect of the zero eigenvalue is clear: There is a mode (namely the zero mode) that does not evolve in time, and since all other modes are decaying, as $t \to \infty$ the solution approaches that mode, i.e., $u(x,t) \to \hat{f}_0$. In our case, the zero mode is simply the average of the initial datum f. Note that there was no trouble as to the solvability of the problem caused by the zero eigenvalue, to be contrasted with the Poisson case.

Finally, let us consider the *wave equation*

$$u_{tt} = \Delta u, \qquad u|_{t=0} = f, \qquad u_t|_{t=0} = g,$$

with the homogeneus Neumann boundary condition. Expanding the solution as in the previous paragraph, we have

$$\hat{u}_k'' = \lambda_k \hat{u}_k,$$

whose solution is given by

$$\hat{u}_0(t) = A_0 + B_0 t, \qquad \hat{u}_k(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t) \quad \text{for} \quad k = 1, 2, \ldots,$$

where $\omega_k = \sqrt{-\lambda_k}$. Now we put together the individual modes of u to get

$$u(x,t) = A_0 + B_0 t + \sum_{k=1}^{\infty} \left(A_k \cos(\omega_k t) + B_k \sin(\omega_k t)\right) v_k(x),$$

where we have taken into account that $v_0 = 1$. From the initial conditions we have $A_k = \hat{f}_k$ for all k, and $B_0 = \hat{g}_0$ and $B_k = \hat{g}_k/\omega_k$ for nonzero k, leading to

$$u(x,t) = \hat{f}_0 + \hat{g}_0 t + \sum_{k=1}^{\infty} \left(\hat{f}_k \cos(\omega_k t) + \frac{\hat{g}_k}{\omega_k} \sin(\omega_k t) \right) v_k(x).$$

The effect of the zero eigenvalue is that there is a mode (namely the zero mode) that does not oscillate in time. Depending on the average of g, the zero-mode may be evolving linearly in time, and all the other modes oscillate around this mode.