SOLUTIONS TO PROBLEMS 3, 5, AND 9 FROM ASSIGNMENT 1

Problem 3

Statement. Consider the Poisson equation $u_{xx} = f$ on the interval $(0, 1)$ subject to the boundary conditions $u_x(0) = \alpha$ (Neumann), and $u(1) = \beta$ (Dirichlet). Find the solution using Green's function approach.

Solution. The boundary value problem we are trying to solve is

$$u_{xx} = f \quad \text{in} \quad (0,1), \qquad u_x(0) = \alpha, \qquad u(1) = \beta. \tag{1}$$

Following the Green function philosophy, we suppose that the solution u, depending on the boundary conditions α, β, and the right hand side f, has the form

$$u(x) = \alpha G_0(x) + \beta G_1(x) + \int_0^1 G(x,t) f(t) \mathrm{d}t. \tag{2}$$

What gives this approach its power is that the Green functions $G_0(x)$, $G_1(x)$ and $G(x,t)$ are the *same* no matter what values α, β, and f have for the particular instance of the boundary value problem (1). So once you have G_0, G_1 and G, you can solve the problem for every value of α, β, and f by using (2), moreover, the functions G_0, G_1 and G themselves can be found by solving the problem for carefully chosen data α, β, and f. For example, we see from (2) that G_0 would be the solution of the problem (1) if we had $\alpha = 1$, $\beta = 0$, and $f = 0$. Similarly, G_1 is the solution if $\alpha = 0$, $\beta = 1$, and $f = 0$. Noting that $f = 0$ means $u_{xx} = 0$, hence that u is a linear function, these solutions are easy to compute:

- For $\alpha = 1$, $\beta = 0$, and $f = 0$, the solution is $u(x) = x - 1$, hence $G_0(x) = x - 1$.
- For $\alpha = 0$, $\beta = 1$, and $f = 0$, the solution is $u(x) = 1$, hence $G_1(x) = 1$.

At this point, let us record our progress:

$$u(x) = \alpha(x-1) + \beta + \int_0^1 G(x,t) f(t) \mathrm{d}t, \tag{3}$$

and note that our aim now is to find $G(x,t)$. To move forward, observe that α and β were just numbers (as opposed f, which is a function), and we found the Green function G_0 by switching off all the inputs except α (similarly for G_1 and β). The situation with f is a bit complex because it is a function. Nevertheless, if you fix t between 0 and 1, then $f(t)$ is just a plain old number, meaning that we can treat f as an infinite collection of numbers, with one number for each $t \in (0,1)$. So an idea would be to fix some $z \in (0,1)$, and consider such an f that $f(z) = 1$ and $f(t) = 0$ for all t not equal to z. But we quickly discover that this would not work, because f enters in (3) through an integral only, and since f is zero everywhere except only at the point z, the integral evaluates to zero. The trick is to introduce the so called *delta "function"* $\delta(t)$, which is characterized by the properties that $\delta(t) = 0$ for all $t \neq 0$, and

$$\int_{-\varepsilon}^{\varepsilon} g(t)\delta(t) \mathrm{d}t = g(0), \tag{4}$$

for $\varepsilon > 0$ and for all continuous function g. Putting $g(t) = 1$ in this formula, we get

$$\int_{-\varepsilon}^{\varepsilon} \delta(t) \mathrm{d}t = 1, \tag{5}$$

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which says that the area under the graph of δ is 1. Since $\delta(t)$ is zero except at $t = 0$, the integral of δ can be nonzero only if $\delta(0) = \infty$, but $\delta(0) = \infty$ alone does not imply (5), because $0 \cdot \infty$ is not defined. So we just leave $\delta(0)$ undefined, and take (4) as part of the definition of δ. The delta "function" is not a function in the ordinary sense, but it can be treated as a function, with the only caveat that the "value" of δ at 0 must be accessed through an integral, as in (4) or (5). One can think of δ as the density of a unit mass concentrated at 0.

Returning back to the problem at hand, in order to find $G(x, z)$ we take the right hand side $f(t)$ to be $\delta(t - z)$, the delta function shifted so that the concentration is at z. Substituting $\alpha = \beta = 0$, and $f(t) = \delta(t - z)$ into (3), we indeed get

$$u(x) = \int_0^1 G(x,t)\delta(t-z)\mathrm{d}t = \int_{-z}^{1-z} G(x, z+s)\delta(s)\mathrm{d}s = G(x,z), \tag{6}$$

where we made the change of variable $s = t - z$, and took into account the fact that the interval $(-z, 1-z)$ contains the point 0 because $0 < z < 1$. To reiterate, (6) says that we should find the solution $u(x)$ of the boundary value problem (1) with $\alpha = \beta = 0$ and $f(t) = \delta(t - z)$, to get $G(x, z)$. To find the solution, first note that $u(x)$ is linear except possibly at $x = z$. Taking note of the boundary conditions $u_x(0) = 0$ and $u(1) = 0$, this means that

$$u(x) = \begin{cases} m & \text{for} \quad x < z, \\ k(x-1) & \text{for} \quad x > z, \end{cases} \tag{7}$$

with some constants m and k to be determined from the behavior of f at z. To extract this information we will employ (5). The integral of f over a small interval around z gives

$$1 = \int_{z-\varepsilon}^{z+\varepsilon} f(x)\mathrm{d}x = \int_{z-\varepsilon}^{z+\varepsilon} u_{xx}(x)\mathrm{d}x = u_x(z+\varepsilon) - u_x(z-\varepsilon) = k, \tag{8}$$

revealing the value of k. Now we know that

$$u_x(x) = \begin{cases} 0 & \text{for} \quad x < z, \\ 1 & \text{for} \quad x > z, \end{cases} \tag{9}$$

and from

$$u(x) = \int_0^x u_x(s)\mathrm{d}s, \tag{10}$$

we conclude that u is continuous at z. In other words, the values of $u(x)$ as x approaches z from the left and right must match. Staring at (7) a bit gives $m = z - 1$, finally uncovering

$$u(x) = \begin{cases} z-1 & \text{for} \quad x \le z, \\ x-1 & \text{for} \quad x \ge z. \end{cases} \tag{11}$$

Obviously, this is also $G(x, z)$:

$$G(x,z) = \begin{cases} z-1 & \text{for} \quad x \le z, \\ x-1 & \text{for} \quad x \ge z. \end{cases} \tag{12}$$

By substituting this into (3), we finally get

$$\begin{aligned} u(x) &= \alpha(x-1) + \beta + \int_0^1 G(x,t)f(t)\mathrm{d}t \\ &= \alpha(x-1) + \beta + \int_0^x (x-1)f(t)\mathrm{d}t + \int_x^1 (t-1)f(t)\mathrm{d}t. \end{aligned} \tag{13}$$

For completeness, note that we derived (13) by using the delta "function", which we do not have a very strict handle on. This means at this level, we cannot be sure that (13) solves (1). But now that we have the expression (13), we can always check explicitly if (13) satisfies the conditions in (1). We can check off the condition $u(1) = \beta$ quickly. For the others, recall the Leibniz rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} g(x,t)\,\mathrm{d}t = \int_{a(x)}^{b(x)} g_x(x,t)\,\mathrm{d}t \; + b_x(x)g(x,b(x)) - a_x(x)g(x,a(x)). \tag{14}$$

From (13), we compute

$$u_x(x) = \alpha + \int_0^x f(t)\mathrm{d}t + 1\cdot(x-1)f(x) + \int_x^1 0\cdot\mathrm{d}t - 1\cdot(x-1)f(x) = \alpha + \int_0^x f(t)\mathrm{d}t,$$

which in particular gives $u_x(0) = \alpha$. Taking the derivative once again, we have

$$u_{xx}(x) = \int_0^x 0\cdot\mathrm{d}t + 1\cdot f(x) = f(x),$$

completing the proof that (13) solves (1).

Problem 5

Statement. Derive the expression for the Laplacian $\Delta u = u_{xx} + u_{yy}$ in polar coordinates. Find all radially symmetric solutions of the Laplace equation $\Delta u = 0$ for $r > 0$ (meaning that $\Delta u = 0$ holds except possibly at the origin).

Solution. We address only the second part of the question. In the polar coordinates (r,ϕ) the Laplacian reads

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = \frac{(ru_r)_r}{r} + \frac{u_{\phi\phi}}{r^2}. \tag{15}$$

Since u is radially symmetric (i.e., u depends only on r), obviously $u_{\phi\phi} = 0$, so the Laplace equation for u becomes

$$\Delta u = \frac{(ru_r)_r}{r} = 0. \tag{16}$$

In the region $r > 0$, we can multiply the preceding equation by r to get

$$(ru_r)_r = 0, \tag{17}$$

which means $ru_r(r) = A$ for some constant A. Therefore

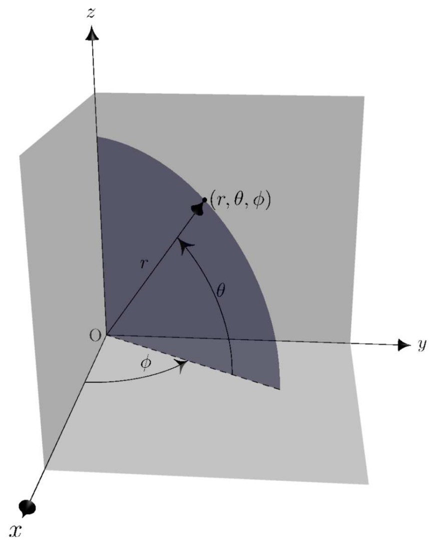
$$u_r(r) = \frac{A}{r}. \tag{18}$$

This has the solution $u(r) = A\log r + B$, where B is a constant. In words, the radially symmetric solutions of the Laplace equation in two dimensions are constants, and if we exclude the point $r = 0$, plus multiples of $\log r$.

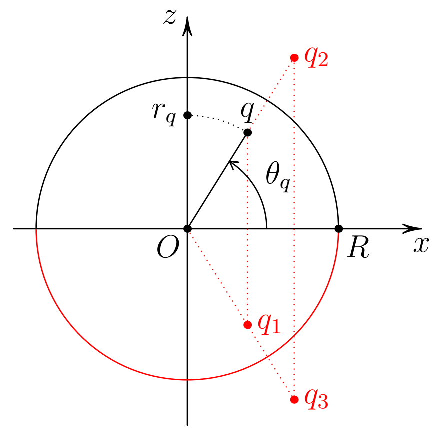
Problem 9

Statement. A cavity in a conductor has the shape of a half sphere, being bounded by the surfaces $\theta = 0$ and $r = R$ in spherical coordinates. A point charge q is located at $\theta = \theta_q$, $r = r_q$, and $\phi = 0$, where $0 < \theta_q \leq \frac{\pi}{2}$ and $0 < r_q < R$.

a) Give the locations and magnitudes of the image charges necessary to maintain the electric potential $\varphi = 0$ at the boundary of the cavity.
b) Write down the potential inside the cavity.



(a) Spherical coordinates



(b) Image charges are q_1, q_2 and q_3.

Solution. In the figures above, we sketch the spherical coordinate system we use, and the configuration of the image charges. Since everything happens in the $\phi = 0$ plane (also known as the xz-plane), the problem is essentially two dimensional.

One needs the image charge q_1 to cancel the potential generated by q on the xy-plane, and q_2 to cancel the potential of q on the surface of the sphere. But q_2 will induce a nonzero potential on the xy-plane, while q_1 induces a nonzero potential on the sphere. To cancel these potential, we need only one additional charge q_3, which is at the same time the reflection of q_2 with respect to the xy-plane, and the reflection of q_1 with respect to the sphere. Note that this is because of the symmetry of this particular geometry, and in general the reflections of two different image charges will not coincide. It is easy to see that if q is at $X = (r_q, \theta_q, 0)$ then q_1 is at $X_1 = (r_q, -\theta_q, 0)$, q_2 is at $X_2 = (\frac{R^2}{r_1}, \theta_q, 0)$, and q_3 is at $X_3 = (\frac{R^2}{r_1}, -\theta_q, 0)$. For the magnitudes, we have $q_1 = -q$, $q_2 = -\frac{R^2}{r_q^2} q$, and $q_3 = \frac{R^2}{r_q^2} q$.

The potential inside the cavity is the sum of the potentials induced by the individual (image and real) charges, so we conclude

$$\varphi(x) = \frac{Cq}{|x - X|} - \frac{Cq}{|x - X_1|} - \frac{CR^2 q}{r_q^2 |x - X_2|} + \frac{CR^2 q}{r_q^2 |x - X_3|}.$$