Lecture 9: Advection in 1D

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Math 319: Introduction to PDEs

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Questions

- What is the solution of $u_t = \lambda u$, with $u(0) = C$?
- $u(t) = Ce^{\lambda t}$.
- What is an orthogonal matrix?
- A such that $A^T A = I$.
- Any symmetric matrix A can be written as $A = PDP^T$ where D is diagonal and P is orthogonal. What are on the diagonal of D?
- The eigenvalues of A.

Advection equation

Let $\Omega = [0,1]^2$ be the unit square, and let g be any function defined on its boundary $\partial\Omega$. We have seen that the following problem has a solution

$$u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega, \qquad \text{and} \qquad u = g \quad \text{on} \quad \partial\Omega.$$

One can say that for the Laplace equation, **every point is influenced by every other point**.

To contrast, consider the **advection equation**

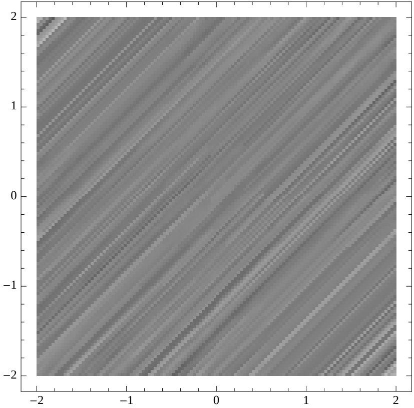
$$u_x + u_y = 0 \qquad \text{in} \quad \Omega.$$

Equivalently, the directional derivative of u along $n = (1,1)$ is zero. So u must be constant along the lines $y = x + C$, or in other words, we have

$$u(x,y) = F(x-y),$$

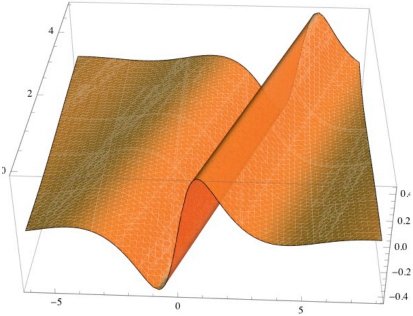
for some function F. In this case, **the points on the line** $y = x + C_1$ **do not care about what is happening on the line** $y = x + C_2$, (if $C_1 \neq C_2$).

The characteristic lines $y = x + C$ of $u_x + u_y = 0$.



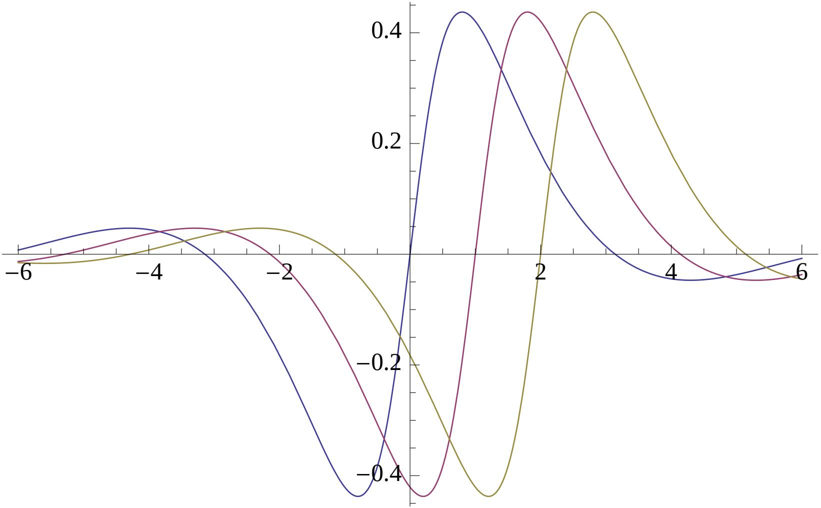
A solution of $u_x + u_y = 0$, where

$$F(x) = \frac{\sin x}{1 + x^2}.$$



Snapshots

Snapshots at $y = 0$, 1, and 2.



Initial value problem

Note that if $u(x,y) = F(x-y)$, then $u(x,0) = F(x)$. A convenient way to specify the "boundary conditions" for the advection equation is to specify the function $u(x,0) = F(x)$. So the solution is obtained by specifying u on the line $y = 0$ only. This "boundary condition" is called **initial condition**, and the variable y is usually associated to time, denoted by t.

The **initial value problem** for the advection equation is

$$u_t + cu_x = 0 \qquad \text{for} \quad t > 0, \qquad \text{and} \qquad u(x,0) = f(x),$$

where c is a constant (speed) and f is a given function.

Obviously, u is constant along the direction $n = (c,1)$ on the (x,t)-plane. In other words, u is constant along the lines $x = ct + K$. Equivalently, u has the form

$$u(x,t) = F(x-ct).$$

The lines $x = ct + K$ are called the **characteristic lines**.

Characteristic coordinates

The **characteristic coordinates** are defined by

$$\xi = x - ct \qquad \text{and} \qquad \tau = t.$$

Let $U(\xi,\tau) = u(\xi + c\tau, \tau)$, or equivalently, $u(x,t) = U(x - ct, t)$. Then

$$u_x = U_\xi, \quad u_t = -cU_\xi + U_\tau, \qquad \text{so} \qquad u_t + cu_x = U_\tau.$$

In the new coordinates, the above problem is simply

$$U_\tau = 0 \qquad \text{for} \quad \tau > 0, \qquad \text{and} \qquad U(\xi, 0) = f(\xi).$$

The solutions is $U(\xi,\tau) = f(\xi)$, and in terms of the "old" variables, we get

$$u(x,t) = U(x - ct, t) = f(x - ct).$$

Damped advection equations

Let us solve

$$u_t + cu_x = \lambda u \qquad \text{for} \quad t > 0, \qquad \text{and} \qquad u(x,0) = f(x).$$

In characteristic coordinates, it is

$$U_\tau = \lambda U \qquad \text{for} \quad \tau > 0, \qquad \text{and} \qquad U(\xi,0) = f(\xi).$$

This is an ODE, with ξ acting as a parameter, and its solution is

$$U(\xi,\tau) = C(\xi)e^{\lambda\tau}.$$

For $\tau = 0$ we have $U(\xi,0) = f(\xi)$, so $C(\xi) = f(\xi)$, giving $U(\xi,\tau) = f(\xi)e^{\lambda\tau}$. In terms of the (x,t) variables, we have

$$u(x,t) = U(x-ct,t) = f(x-ct)e^{\lambda t}.$$

Symmetric hyperbolic systems

Let A be a symmetric $n \times n$ matrix, and we want to solve

$$u_t + Au_x = 0,$$

for the unknown function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$. We can write $A = PDP^T$ with a diagonal matrix D, and an orthogonal matrix P, meaning that $P^T P = I$. The above equation can be written as

$$u_t + PDP^T u_x = 0,$$

and after multiplying by P^T from the left, it becomes

$$P^T u_t + DP^T u_x = 0.$$

Upon introducing $q = P^T u$, the new variable q satisfies

$$q_t + Dq_x = 0.$$

This is just n separate advection equations for the components of q.