Lecture 8: Qualitative study of the Dirichlet problem

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- Let $E(x)$ be a function of one variable x. For example, $E(x) = x^4 - 3x$. How would you find a minimum of E?
- Solve $E_x(x) = 0$.
- Suppose that x_* satisfies $E_x(x_*) = 0$. Does this mean that x_* is a minimum of E?
- No. It maybe a maximum, or a more general critical point.
- Consider $E(x) = 1/x$ on the positive reals $x > 0$. Obviously $E(x) \geq 0$. Does E have a minimum? I.e., is there a point $x_* > 0$ such that $E(x_*) \leq E(x)$ for all $x > 0$?
- No.

Well-posedness

For a domain $\Omega \subset \mathbb{R}^3$, and a function g defined on the boundary $\partial\Omega$, we consider the *Dirichlet problem for the Laplace equation*

$$\begin{aligned} -\Delta u = 0 &\quad \text{in} \quad \Omega, \\ u = g &\quad \text{on} \quad \partial\Omega. \end{aligned}$$

Well-posedness study addresses the following questions:

- Does there exist a solution u?
- How many solutions? Is it unique?
- How sensitive does the solution depend on the data?

If there is a unique solution, and it depends on the data continuously, then we say that the problem is **well-posed**. Well-posed problems are ideal for modeling, if we want to predict and possibly control the system.

Define the **Dirichlet energy**

$$\mathcal{E}(w) = \int_\Omega |\nabla w|^2 = \int_{\partial\Omega} w\nabla w - \int_\Omega u\Delta w,$$

so we have

$$\mathcal{E}(u) = \int_{\partial\Omega} g\nabla u.$$

We have

$$\begin{aligned}\mathcal{E}(u+v) &= \int_\Omega |\nabla u|^2 + 2\int_\Omega \nabla u\cdot\nabla v + \int_\Omega |\nabla v|^2 \\ &= \mathcal{E}(u) + 2\int_{\partial\Omega} v\nabla u - 2\int_\Omega v\Delta u + \mathcal{E}(v) \\ &= \mathcal{E}(u) + \mathcal{E}(v),\end{aligned}$$

for all v such that $v = 0$ at the boundary $\partial\Omega$.

> *Hence the solution u minimizes the Dirichlet energy over all functions with the boundary condition $u = g$ on $\partial\Omega$.*

On the other hand, if u is a minimizer of the Dirichlet energy, then we have, for arbitrary v with $v = 0$ on $\partial\Omega$

$$\mathscr{E}(u+\varepsilon v) = \mathscr{E}(u) - 2\varepsilon \int_\Omega v\Delta u + \varepsilon^2 \mathscr{E}(v) \qquad \Rightarrow \qquad \int_\Omega v\Delta u = 0.$$

So finding a minimizer of $\mathscr{E}$ gives a solution to the Dirichlet problem.

It is known that a minimizer exists provided the boundary of the domain Ω is not too "wild". The justification is a subject of a second course on PDE. For the interested, a simpler method for obtaining existence of solutions to the Dirichlet problem is the so called *Perron's method*.

Uniqueness by energy method

Recall that if u is a solution of the Dirichlet problem then

$$\mathscr{E}(u+v) = \mathscr{E}(u) + \mathscr{E}(v),$$

for all v such that $v = 0$ at the boundary $\partial\Omega$.

Let w be another solution of the Dirichlet problem. Then, setting $v = w - u$, we have

$$\mathscr{E}(w) = \mathscr{E}(u) + \mathscr{E}(w-u).$$

By interchanging the roles of u and w, we get

$$\mathscr{E}(u) = \mathscr{E}(w) + \mathscr{E}(u-w).$$

Hence

$$\mathscr{E}(u-w) = \int_\Omega |\nabla(u-w)|^2 = 0,$$

implying that $u - w = \text{constant}$, but $u - w = 0$ at the boundary, so

$$u \equiv w.$$

Maximum principles for harmonic functions

Let u be a harmonic function, defined on a bounded domain Ω.

First of all, we have the **weak maximum principle**:

$$u(x) \leq \max_{y \in \partial\Omega} u(y) \qquad \text{for any} \qquad x \in \Omega.$$

Furthermore, the **strong maximum principle** holds:

> *If $u(x) \leq \max_{y \in \partial\Omega} u(y)$ for some $x \in \Omega$, i.e., if u attains its maximum at an interior point, then u must be a constant function.*

Uniqueness by maximum principle: If u and w are both solutions of the Dirichlet problem, we have

$$\begin{aligned} -\Delta(u-w) &= 0 \quad \text{in} \quad \Omega, \\ u-w &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \qquad \begin{aligned} -\Delta(w-u) &= 0 \quad \text{in} \quad \Omega, \\ w-u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

By applying the weak maximum principle to $u-w$ and $w-u$, we conculde that $u = w$.

Dirichlet problem for the Poisson equation

The Dirichlet principle and the uniqueness arguments can be adapted to the following Dirichlet boundary value problem

$$
\begin{aligned}
-\Delta u = f \qquad &\text{in} \quad \Omega, \\
u = g \qquad &\text{on} \quad \partial\Omega.
\end{aligned}
$$

In this case, the Dirichlet energy would be

$$\mathscr{E}(w) = \int_\Omega |\nabla w|^2 - \int_\Omega fw.$$