Lecture 7: Finite difference method

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5-point formula

Consider the Dirichlet problem for the 2-dimensional Poisson equation

$$\begin{aligned} -\Delta u = f \qquad & \text{in} \quad \Omega, \\ u = g \qquad & \text{on} \quad \partial\Omega. \end{aligned}$$

The Laplacian can be approximated by the **5-point formula**

$$-\Delta u(x) \approx \frac{4u(x,y) - u(x+h,y) - u(x-h,y) - u(x,y+h) - u(x,y-h)}{h^2}.$$

Suppose $u(ih,kh) \approx u_{i,k}$, and at the interior nodes we require

$$4u_{i,k} - u_{i+1,k} - u_{i-1,k} - u_{i,k+1} - u_{i,k-1} = f_{ik} := h^2 f(ih,ik).$$

At the boundary nodes (i,k), we impose

$$u_{i,k} = g_{ik} := g(ih,ik).$$

So each unknown $u_{i,k}$ has its corresponding equation.

Discrete mean value property

If $f = 0$, i.e., if u is **harmonic**, then the corresponding approximate solution satisfies

$$4u_{i,k} - u_{i+1,k} - u_{i-1,k} - u_{i,k+1} - u_{i,k-1} = 0,$$

or equivalently

$$u_{i,k} = \frac{u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1}}{4}.$$

So $u_{i,k}$ is the average of the values at the 4 neighbouring nodes.

Consider $u_{i,k}$ as a function sending the node (i,k) to the value $u_{i,k}$.

Theorem

If $u_{i,k}$ attains its maximum at an interior node, $u_{i,k}$ must be constant.

Proof.

We have the mean value property

$$u_{i,k} = \frac{u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1}}{4}.$$

Suppose that $M = \max_{i,k} u_{i,k}$ and $M = u_{i,k}$ for some particular (interior) node (i,k). Then it must be the case that

$$u_{i+1,k} = u_{i-1,k} = u_{i,k+1} = u_{i,k-1} = M,$$

otherwise they cannot have M as their average. This reasoning "propagates" to make every $u_{i,k} = M$. □