Lecture 6: Dirichlet problem

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Dirichlet problem for the Poisson equation

Let $\Omega \subset \mathbb{R}^3$ be a domain, and let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, where $\partial\Omega$ is the boundary of Ω. The *Dirichlet problem* is: Find $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{aligned} -\Delta u = f \qquad & \text{in} \quad \Omega, \\ u = g \qquad & \text{on} \quad \partial\Omega. \end{aligned}$$

For example, the electric potential φ of a charge distribution ρ inside a cavity Ω with grounded conductor boundary $\partial\Omega$, satisfies

$$-\Delta\varphi = 4\pi C\rho \qquad \text{in} \quad \Omega, \qquad \text{and} \qquad \varphi = 0 \qquad \text{on} \quad \partial\Omega.$$

If $\rho(x) = Q\delta(x)$ and Ω is the ball centered at the origin, of radius R, then

$$\varphi(x) = \frac{CQ}{|x|} - \frac{CQ}{R}.$$

Suppose that we can find v satisfying $-\Delta v = f$ in Ω (and not necessarily $v = g$ on $\partial\Omega$). If we look for u in the form $u = v + h$, then h must satisfy the *Dirichlet problem for the Laplace equation*

$$\begin{aligned} -\Delta h &= 0 \qquad \text{in} \quad \Omega, \\ h &= g - v \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

Example: Suppose that the entire half space $\{x_1 \leq 0\}$ is filled with perfect conductor, and a charge Q is sitting at $X = (a, 0, 0)$ with $a > 0$. The potential u must satisfy $-\Delta u(x) = 4\pi CQ\delta(x - X)$ for $x_1 > 0$ and $u(x) = 0$ for $x_1 = 0$. We know that

$$v(x) = \frac{CQ}{|x - X|} \qquad \text{satisfies} \qquad -\Delta v(x) = 4\pi CQ\delta(x - X).$$

In order to have $u = v + h = 0$ at $x_1 = 0$, we take

$$h(x) = -\frac{CQ}{|x - X^*|}, \qquad \text{with} \qquad X^* = (-a, 0, 0).$$

Spherical conductors

Now consider a grounded conducting sphere of radius R, with center at $x=0$, and suppose that there is a charge Q at $X=(a,0,0)$ with $0\leq a<R$. We want to find X^* and Q^* such that

$$\frac{CQ}{|x-X|}+\frac{CQ^*}{|x-X^*|}=0,$$

at the spherical surface. We can rearrange this as

$$Q^*=-Q\cdot\frac{|x-X^*|}{|x-X|},$$

so the ratio in the right hand side cannot depend on x anywhere on the spherical surface. From symmetry, we have $X^*=(b,0,0)$ for some $b>R$. This leads to

$$Q^*=-\frac{R}{a}Q, \qquad \text{and} \qquad b=\frac{R^2}{a}.$$

Finite differences

Consider the Dirichlet problem for the 2-dimensional Poisson equation

$$\begin{aligned} \Delta u = f \qquad & \text{in} \quad [0,1]^2, \\ u = 0 \qquad & \text{on} \quad \partial[0,1]^2. \end{aligned}$$

A straightforward way to approximately solve this equation on a computer is to use *finite differences*. We cover the square with a *mesh* of squares of size $h = 1/n$ for some integer n. With integers i and k, the points (hi, hk) are called *nodes* of the mesh.
The Laplacian can be approximated by the 5-point formula

$$\Delta u(x) \approx \frac{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)}{h^2},$$

which leads to a linear system of equations for the approximate values of u at the nodes.