Lecture 4: Poisson equations, electrostatics

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Math 319: Introduction to PDEs

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January 11, 2011

The Dirac "function" δ satisfies

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0, \qquad \text{and} \qquad \int \delta(x)\mathrm{d}x = 1.$$

One can think of δ as the density of a unit mass located at $x = 0$.

An important property of δ is (for continuous f)

$$\int \delta(x) f(x)\mathrm{d}x = f(0), \qquad \text{or equivalently,} \qquad \int \delta(x-t) f(x)\mathrm{d}x = f(t).$$

One way to make this idea rigorous is to consider δ as a *linear functional*. A linear functional is a linear operator sending *functions to numbers*. For example, the operation of taking average

$$Af = \int_0^1 f(x)\mathrm{d}x,$$

is a linear functional. The delta functional is simply the operation of evaluating the function at a point: $\delta_t f = f(t)$.

Initial value problem for the Poisson equation in 1D

For the problem $u_{xx} = f$ with initial conditions $u(0) = \alpha$ and $u_x(0) = \beta$, we derived a *candidate* solution

$$u(x) = \alpha + \beta x + \int_0^1 G(x,t) f(t)\,\mathrm{d}t,$$

where $G_{xx}(x,t) = \delta(x-t)$, and $G(0,t) = G_x(0,t) = 0$. More explicitly,

$$G(x,t) = \begin{cases} 0 & \text{for} \quad x \leq t, \\ x-t & \text{for} \quad x \geq t. \end{cases}$$

So we can write

$$u(x) = \alpha + \beta x + \int_0^x (x-t) f(t)\,\mathrm{d}t.$$

Notice that $u(0) = \alpha$.

Recall the *Leibniz rule* (for g such that g and g_x are continuous)

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} g(x,t)\,\mathrm{d}t = \int_{a(x)}^{b(x)} g_x(x,t)\,\mathrm{d}t \; + b_x(x)g(x,b(x)) - a_x(x)g(x,a(x)).$$

Our proposed solution is

$$u(x) = \alpha + \beta x + \int_0^x (x-t)f(t)\mathrm{d}t.$$

Differentiating

$$u_x(x) = \beta + \int_0^x f(t)\mathrm{d}t + 1\cdot(x-x)f(x) \qquad \Rightarrow \qquad u_x(0) = \beta.$$

Differentiating once more

$$u_{xx}(x) = \int_0^x 0\cdot\mathrm{d}t + 1\cdot f(x) \qquad \Rightarrow \qquad u_{xx} = f.$$

Boundary value problem for the Poisson equation

For the problem $u_{xx} = f$ with boundary conditions $u(0) = \alpha$ and $u(1) = \beta$, we derived a *candidate* solution

$$u(x) = \alpha(1-x) + \beta x + \int_0^1 G(x,t) f(t)\,\mathrm{d}t.$$

where

$$G(x,t) = \begin{cases} (t-1)x & \text{for} \quad x \leq t, \\ t(x-1) & \text{for} \quad x \geq t. \end{cases}$$

We can write

$$u(x) = \alpha(1-x) + \beta x + \int_0^x (x-1)\,t f(t)\,\mathrm{d}t + \int_x^1 x(t-1) f(t)\,\mathrm{d}t.$$

Note that $u(0) = \alpha$ and $u(1) = \beta$.

Recall the Leibniz rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} g(x,t)\,\mathrm{d}t = \int_{a(x)}^{b(x)} g_x(x,t)\,\mathrm{d}t \; + b_x(x)g(x,b(x)) - a_x(x)g(x,a(x)),$$

and the proposed solution

$$u(x) = \alpha(1-x) + \beta x + \int_0^x (x-1)tf(t)\mathrm{d}t + \int_x^1 x(t-1)f(t)\mathrm{d}t.$$

We compute

$$u_x(x) = -\alpha + \beta + \int_0^x tf(t)\mathrm{d}t + 1\cdot(x-1)xf(x) + \int_x^1 (t-1)f(t)\mathrm{d}t - 1\cdot x(x-1)f(x),$$

and finally

$$u_{xx}(x) = 0 + 1\cdot xf(x) - 1\cdot(x-1)f(x) \qquad \Rightarrow \qquad u_{xx} = f.$$

Let q and Q be electric charges with locations given by the vectors $x \in \mathbb{R}^3$ and $X \in \mathbb{R}^3$, respectively. Then by *Coulomb's law*, the electrostatic force exerted on q is equal to

$$F = \frac{CQq}{|x-X|^2} \cdot \frac{x-X}{|x-X|},$$

where C is a constant that depends on the unit system.
If several charges Q_i are present at R_i, then by the *superposition principle*, the net force acting on q is

$$F = q\sum_i \frac{CQ_i}{|x-X_i|^2} \cdot \frac{x-X_i}{|x-X_i|}.$$

It is convenient to identify the electric field generated by the charges Q_i with the vector-valued function

$$E(x) = \sum_i \frac{CQ_i}{|x-X_i|^2} \cdot \frac{x-X_i}{|x-X_i|}.$$

Observe that

$$\frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2}x_1 = -\frac{3}{2}\left(|x|^2\right)^{-5/2}\cdot 2x_1\cdot x_1 + \left(|x|^2\right)^{-3/2} = \frac{-2x_1^2+x_2^2+x_3^2}{|x|^5}$$

and so

$$\begin{aligned}\nabla\cdot\frac{x}{|x|^3} &= \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2}x_1 + \frac{\partial}{\partial x_2}\left(|x|^2\right)^{-3/2}x_2 + \frac{\partial}{\partial x_3}\left(|x|^2\right)^{-3/2}x_3\\ &= \frac{-2x_1^2+x_2^2+x_3^2}{|x|^5} + \frac{-2x_2^2+x_1^2+x_3^2}{|x|^5} + \frac{-2x_3^2+x_1^2+x_2^2}{|x|^5} = 0.\end{aligned}$$

Therefore

$$\nabla\cdot E = 0, \qquad \text{except at} \quad X_i.$$

Observe also that

$$\frac{\partial}{\partial x_1}\frac{1}{|x|} = \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-1/2} = -\frac{1}{2}\left(|x|^2\right)^{-3/2}\cdot 2x_1 = -\frac{1}{|x|^2}\cdot\frac{x_1}{|x|},$$

and so

$$\nabla\frac{1}{|x|} = -\frac{1}{|x|^2}\cdot\frac{x}{|x|}.$$

Define the *electric potential*

$$\varphi(x) = \sum_i \frac{CQ_i}{|x-X_i|}.$$

Then

$$E = -\nabla\varphi, \quad \text{and} \quad \Delta\varphi = 0, \qquad \text{except at} \quad X_i.$$