Lectures 32–34: Examples in separation of variables

Gantumur Tsogtgerel

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McGill University, Montréal

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Radial case, half disk

In the **radial case**, the data and the solution do not depend on ϕ, so e.g., the eigenfunction expansion of f reduces to

$$f(r) = \sum_{k=1}^{\infty} \beta_k J_0(\alpha_{0,k} r),$$

with the coefficients

$$\beta_k = \frac{1}{\pi |J_1(\alpha_{0,k})|^2} \int_0^1 f(r) J_0(\alpha_{0,k} r)\, r \,\mathrm{d}r.$$

Consider the Laplace eigenproblem with the homogeneous Dirichlet condition on the **half disk** $\{(r,\phi) : 0 < r < 1, 0 < \phi < \pi\}$. In the ϕ-direction one has the expansion in terms of $\sin(n\phi)$, with the eigenvalues $-n^2$. This immediately leads to the "pre-Bessel" equation

$$\omega_n''(r) + \frac{1}{r}\omega_n'(r) + \left(\lambda - \frac{n^2}{r^2}\right)\omega_n(r) = 0,$$

implying that the eigenfunctions are $v_{n,k}(r,\phi) = J_n(\alpha_{n,k}) \sin(n\phi)$ with the eigenvalues $-\alpha_{n,k}^2$, where $n = 1, 2, \ldots$.

Exterior Dirichlet problem for the circle

Let us solve the **exterior** Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \mathbb{D}, \qquad u = g \quad \text{on} \quad \partial\mathbb{D}.$$

For $r > 1$, we can write

$$u(r,\phi) = \xi_0(r) + \sum_{n=1}^{\infty} \left(\xi_n(r) \cos n\phi + \xi_{-n}(r) \sin n\phi \right),$$

and

$$g(\phi) = \gamma_0 + \sum_{n=1}^{\infty} \left(\gamma_n \cos n\phi + \gamma_{-n} \sin n\phi \right).$$

We must require $\xi_n(1) = \gamma_n$ for all $n \in \mathbb{Z}$. The ODE for ξ_n is

$$(\xi_n)_{rr} + \frac{1}{r}(\xi_n)_r - \frac{n^2}{r^2}\xi_n = 0,$$

whose only solutions that *do not blow up at* ∞ are $\xi_n(r) = \gamma_n r^{-n}$, giving

$$u(r,\phi) = \gamma_0 + \sum_{n=1}^{\infty} r^{-n} \left(\gamma_n \cos n\phi + \gamma_{-n} \sin n\phi \right) = \int_{-\pi}^{\pi} g(\phi - \theta) P_r(\theta) \mathrm{d}\theta,$$

where $P_r(x) = \frac{1}{2\pi} \cdot \frac{r^2 - 1}{r^2 - 2r\cos x + 1}$ is the *Poisson kernel* for the disk exterior.

Solid cylinder

Consider the eigenvalue problem

$$\Delta u = \lambda u,$$

in the solid cylinder $\mathbb{D} \times (0, \pi)$, with the homogeneous Dirichlet boundary condition. Letting Δ_2 be the 2-dimensional Laplacian, and writing $u(r, \phi, z) = V(r, \phi)Z(z)$ in the cylindrical coordinates, we have

$$\Delta u = Z\Delta_2 V + VZ'' = \lambda VZ.$$

We divide by VZ and rearrange to get

$$\frac{\Delta_2 V}{V} + \frac{Z''}{Z} = \lambda \qquad \Rightarrow \qquad \lambda - \frac{\Delta_2 V}{V} = \frac{Z''}{Z} = -m^2,$$

for any positive integer m, with $Z(z) = \sin(mz)$. Then V must satisfy

$$\Delta_2 V = (\lambda + m^2)V,$$

on the unit disk $\mathbb{D}$, with homogeneous Dirichlet condition. This means

$$\lambda + m^2 = -\alpha_{n,k}^2 \qquad \Rightarrow \qquad \lambda_{n,k,m} = -\alpha_{n,k}^2 - m^2,$$

where $\alpha_{n,k}$ are the positive zeroes of J_n.

Spherically symmetric eigenproblem

The spherically symmetric eigenproblem in the unit ball with the homogeneous Dirichlet condition is

$$\Delta u = u_{rr} + \frac{2}{r} u_r = \lambda u, \qquad u(1) = 0.$$

This can be solved by observing

$$(ru)_{rr} = ru_{rr} + 2u_r,$$

so that the eigenproblem can be written as

$$(ru)_{rr} = \lambda ru.$$

Its only bounded solutions are multiples of

$$u(r) = \frac{\sin(\pi n r)}{r}.$$