Lecture 31: Problems on the disk

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Math 319: Introduction to PDE
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Tuesday March 22, 2011

Laplace eigenfunction expansion

We found that the Laplace **eigenfunctions** on the unit disk with the homogeneous Dirichlet boundary condition are

$$v_{n,k}(r,\phi) = J_n\left(\alpha_{n,k} r\right)\cos n\phi, \quad \text{and} \quad v_{-n,k}(r,\phi) = J_n\left(\alpha_{n,k} r\right)\sin n\phi,$$

with the corresponding **eigenvalues**

$$\lambda_{n,k} = \lambda_{-n,k} = -\alpha_{n,k}^2, \qquad \text{for} \qquad n \geq 0, \quad k \geq 1,$$

where $\alpha_{n,1}, \alpha_{n,2}$, etc, are the positive **zeroes of the Bessel function** J_n. So we have the **eigenpairs** $\{v_{n,k}, \lambda_{n,k}\}$ with $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

We can write any function f with $\|f\| < \infty$, defined on $\mathbb{D}$ as

$$f = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \beta_{n,k} v_{n,k}.$$

The $v_{n,k}$ are pairwise orthogonal w.r.t. the L^2-inner product on $\mathbb{D}$, so

$$\beta_{n,k} \int_{\mathbb{D}} |v_{n,k}|^2 = \int_{\mathbb{D}} f v_{n,k}.$$

Orthogonality of the eigenfunctions

It is a general fact that eigenfunctions corresponding to different eigenvalues are pairwise orthogonal. Suppose, on some domain Ω,

$$\Delta u = \lambda u, \qquad \text{and} \qquad \Delta v = \mu v,$$

with the homogeneous Dirichlet boundary condition. Multiply the first equation by v and integrate to find

$$\lambda \int_\Omega vu = \int_\Omega v\Delta u = \int_{\partial\Omega} v\partial_n u - \int_\Omega \nabla v \cdot \nabla u = -\int_\Omega \nabla v \cdot \nabla u.$$

Similarly, we can manipulate the second equation by multiplying it by u. Then taking the difference of the resulting two expressions, we get

$$(\lambda - \mu)\int_\Omega vu = 0 \qquad \Rightarrow \qquad \int_\Omega vu = 0 \qquad \text{if} \quad \lambda \neq \mu,$$

which is the L^2-**orthogonality** of u and v. The same argument works for the homogeneous Neumann and Robin boundary conditions.

Normalizing constants

To compute the coefficients of the eigenfunction expansions on the disk, we need the values

$$\int_{\mathbb{D}} |v_{n,k}|^2 = \int_0^1 \int_{-\pi}^{\pi} J_n^2(\alpha_{n,k} r) \cos^2(n\phi) r \,\mathrm{d}r \,\mathrm{d}\phi = \pi \left(1 + \delta_{n,0}\right) \int_0^1 J_n^2(\alpha_{n,k} r) r \,\mathrm{d}r.$$

Although the intermediate step (with cosine) is only valid for the case $n \geq 0$, the result is true in general. To compute the integral, we start with the Bessel equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0,$$

and multiply it by $2y'$ to arrive at

$$\left((xy')^2 + (x^2 - n^2) y^2\right)' = 2xy^2.$$

This implies

$$\int_0^1 J_n^2(\alpha_{n,k} r) r \,\mathrm{d}r = \frac{1}{\alpha_{n,k}^2} \int_0^{\alpha_{n,k}} J_n^2(x) x \,\mathrm{d}x = \frac{|J_n'(\alpha_{n,k})|^2}{2} = \frac{|J_{n+1}(\alpha_{n,k})|^2}{2}.$$

Let f and g be functions defined on $\mathbb{D}$. Then with homogeneous Dirichlet boundary conditions, consider

- The Poisson problem $\Delta u = f$
- The heat equation $u_t = \Delta u$, with $u(r,\phi,0) = f(r,\phi)$
- Wave $u_{tt} = \Delta u$, with $u(r,\phi,0) = f(r,\phi)$ and $u_t(r,\phi,0) = g(r,\phi)$

We can write

$$u(r,\phi,t) = \sum_{n\in\mathbb{Z},k\in\mathbb{N}} \xi_{nk}(t) v_{nk}(r,\phi), \quad f = \sum_{n\in\mathbb{Z},k\in\mathbb{N}} \beta_{nk} v_{nk}, \quad g = \sum_{n\in\mathbb{Z},k\in\mathbb{N}} \gamma_{nk} v_{nk},$$

with u (and so ξ_{nk}) not depending on t for the Poisson case. Then

- for Poisson $\xi_{nk} = -\beta_{nk}/\alpha_{nk}^2$
- for heat $\xi_{nk}(t) = e^{-\alpha_{nk}^2 t} \beta_{nk}$
- for wave $\xi_{nk}(t) = \beta_{nk} \cos \alpha_{nk} t + \frac{\gamma_{nk}}{\alpha_{nk}} \sin \alpha_{nk} t$