Lecture 30: Bessel functions and the Laplace eigenfunctions on the disk

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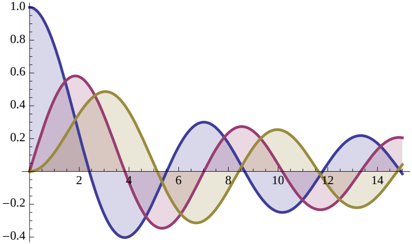
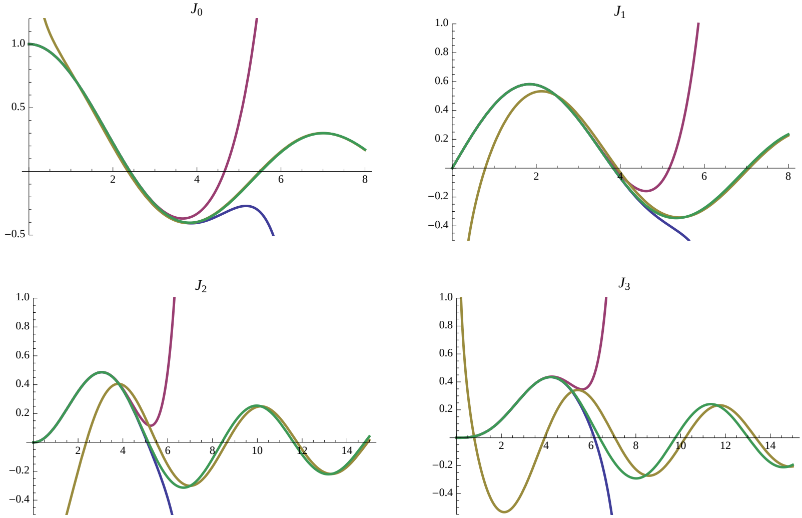
$$\text{Recall } J_n(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!(n+q)!} \left(\frac{x}{2}\right)^{n+2q}.$$

- $J_n(-x) = (-1)^n J_n(x)$
- $J_n(0) = \ldots = J_n^{(n-1)}(0) = 0$, and no positive zeroes of J_n are repeated
- $\left(x^n J_n(x)\right)' = x^n J_{n-1}(x)$, and $\left(x^{-n} J_n(x)\right)' = -x^{-n} J_{n+1}(x)$
- The zeroes of J_n and J_{n+1} are interlaced, and no two J_n and J_k have common zeroes
- $J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\pi n}{2}\right) + O\left(x^{-3/2}\right)$ for large x

J_0, J_1, and J_2

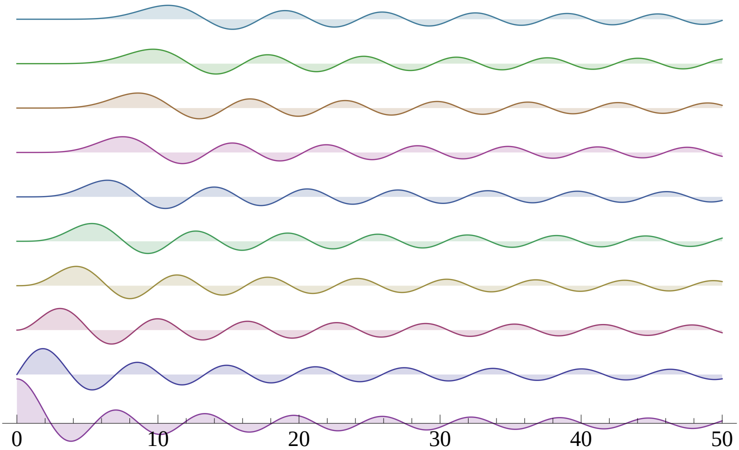


Truncated power series and asymptotic formula



green: J_n, greenish yellow: asymptotic formula, magenta: power series with $q \leq 4$, blue: power series with $q \leq 5$

Bessel zeroes



Laplace eigenfunctions on the disk

Recall that we started with the Laplace eigenproblem on the unit disk with the homogeneous Dirichlet boundary condition. We derived that if

$$v(r,\phi) = \omega_0(r) + \sum_{n=1}^{\infty} \left(\omega_n(r)\cos n\phi + \omega_{-n}(r)\sin n\phi\right),$$

is an eigenfunction with eigenvalue λ, then the function $y(x) = \omega_n(x/\alpha)$ with $\alpha = \sqrt{-\lambda}$, must satisfy the Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

We need $\omega_n(0)$ finite, so

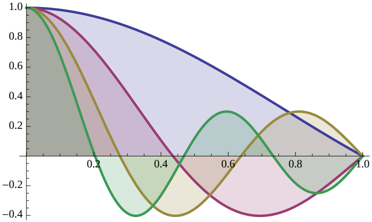
$$\omega_n(r) = J_n(\alpha r).$$

Now the boundary condition $\omega_n(1) = 0$ requires $J_n(\alpha) = 0$, i.e., α must be a zero of J_n. Denoting the positive zeroes of J_n by $\alpha_{n,1}, \alpha_{n,2}$, etc, we have

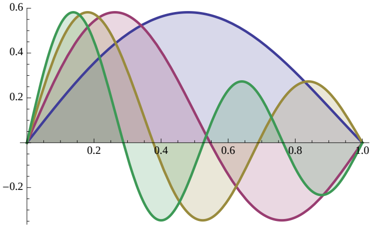
$$v_{n,k}(r,\phi) = J_n\left(\alpha_{n,k}r\right)\cos n\phi, \quad \text{and} \quad v_{-n,k}(r,\phi) = J_n\left(\alpha_{n,k}r\right)\sin n\phi,$$

are the **eigenfunctions** with the **eigenvalue** $-\alpha_{n,k}^2$, for $n \geq 0$ and $k \geq 1$.

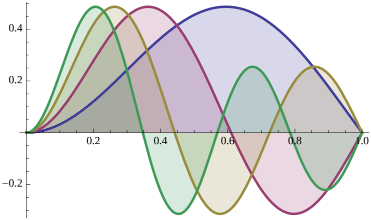
Radial components of the eigenfunctions



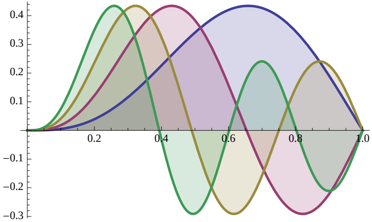
$J_0(\alpha_{0,k}x)$ couples to constants



$J_1(\alpha_{1,k}x)$ couples to $\sin\phi$, $\cos\phi$

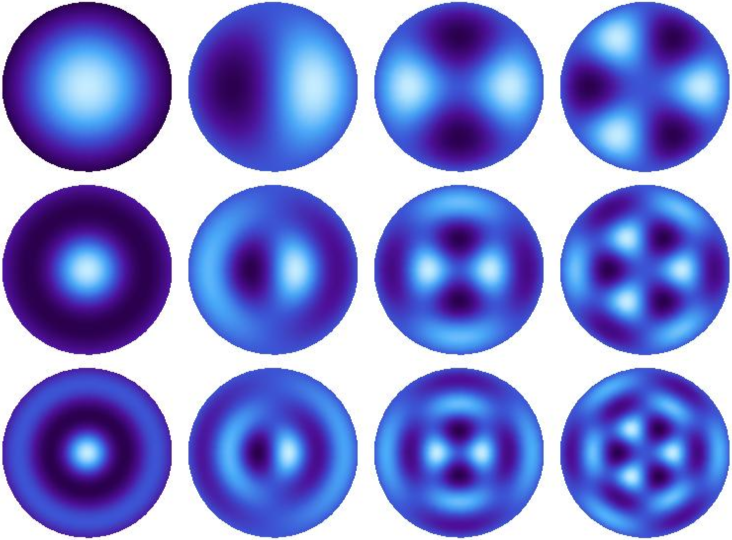


$J_2(\alpha_{2,k}x)$ couples to $\sin 2\phi$, $\cos 2\phi$



$J_3(\alpha_{3,k}x)$ couples to $\sin 3\phi$, $\cos 3\phi$

Laplace eigenfunctions on the disk



Laplace eigenfunctions on the disk