Lecture 3: Initial and boundary value problems in 1 dimension

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Euler's finite difference method

We have $u_x(x) \approx \dfrac{u(x+h)-u(x)}{h}$ for small h, if e.g. $u \in C^1(\mathbb{R})$. So $u_x = f$ is something like

$$y_{i+1} - y_i = b_{i+1},$$

where $y_i \approx u(ih)$ and $b_{i+1} \approx hf(ih)$. This can be solved as

$$y_n = y_{n-1} + b_n = y_{n-2} + b_{n-1} + b_n = \ldots = y_0 + b_1 + \ldots + b_n.$$

Let us rewrite the equation as

$$Ay = b,$$

with

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{n\times(n+1)}.$$

The dimension of $\operatorname{Ker} A$ is 1.

Discrete Green's function

So if we give $y_0 = \alpha$, then the solution is unique. This means we consider the equation $\tilde{A}y = \tilde{b}$ where

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{(n+1)\times(n+1)} \qquad \text{and} \qquad \tilde{b} = \begin{pmatrix} \alpha \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

In particular $\tilde{A} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is an invertible matrix, with the inverse

$$G := \tilde{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

Discrete Green's function

The rows of G correspond to how particular y_i depends on b_k (and α):

$$y_i = 1\cdot\alpha + 1\cdot b_1 + \ldots + 1\cdot b_i + 0\cdot b_{i+1} + \ldots + 0\cdot b_n.$$

The k-th column of G corresponds to the solution of the problem $\tilde{A}y = e_k$ where $e_k = [0,\ldots,0,1,0\ldots,0]$ with the 1 at k-th place. Let $G_0, G_1, \ldots$ be the columns of G. Then we have

$$y = \alpha G_0 + b_1 G_1 + \ldots + b_n G_n.$$

In other words,

- G_0 is the solution when $\alpha = 1$, $b_1 = \ldots = b_n = 0$,
- G_1 is the solution when $\alpha = 0$, $b_1 = 1$, and $b_2 = \ldots = b_n = 0$, etc.

To know a linear system, it is enough to know the responses of the system for a "few" well-chosen set of inputs.

Discrete Green's function

What if $y_n = \beta$ is given, and we want to find $y_0, \ldots, y_{n-1}$? Then we need to "integrate backwards"

$$y_i = y_{i+1} - b_{i+1} = y_{i+2} - b_{i+1} - b_{i+2} = \ldots = y_n - b_{i+1} - \ldots - b_n.$$

Now the problem is $\tilde{\tilde{A}}y = \tilde{\tilde{b}}$ where

$$\tilde{\tilde{A}} = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{(n+1)\times(n+1)} \qquad \text{and} \qquad \tilde{\tilde{b}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ \beta \end{pmatrix},$$

and we have

$$\tilde{\tilde{A}}^{-1} = \begin{pmatrix} -1 & \ldots & -1 & -1 & 1 \\ \vdots & \ddots & & & \vdots \\ 0 & \ldots & -1 & -1 & 1 \\ 0 & \ldots & 0 & -1 & 1 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

The analogue of the matrix inversion for $u_x = f$ with $u(0) = \alpha$ is

$$u(x) = \alpha + \int_0^\infty G(x,t) f(t) \mathrm{d}t,$$

where

$$G(x,t) = \begin{cases} 1 & \text{if} \quad t \le x \\ 0 & \text{otherwise} \end{cases}$$

is called **Green's function** for the problem $u_x = f$ on $(0, \infty)$ with the initial condition at $x = 0$.

The function $G_t(x) = G(x,t)$ is the response to the Dirac function $\delta(x-t)$ centered at t.

$$\delta(x-t) = 0 \quad \text{for} \quad x \neq t, \qquad \text{and} \qquad \int \delta(x-t) f(x) \mathrm{d}x = f(t).$$

Green's function

For the problem $u_x = f$ on $(-\infty, 0)$ with the "terminal" condition at $x = 0$, we have

$$u(x) = u(0) - \int_x^0 f(t)\,\mathrm{d}t = u(0) + \int_{-\infty}^0 G(x,t) f(t)\,\mathrm{d}t,$$

so its Green's function would be

$$G(x,t) = \begin{cases} -1 & \text{if} \quad t \geq x \\ 0 & \text{otherwise} \end{cases}$$

The **Laplace operator** is $\Delta u = u_{xx}$ in 1D, $\Delta u = u_{xx} + u_{yy}$ in 2D etc. The two basic equations involving this operator are the **Laplace equation**

$$\Delta u = 0,$$

whose solutions are called **harmonic functions**, and the more general **Poisson equation**

$$\Delta u = f.$$

Poisson equation in 1D

Consider $u_{xx} = f$ on the interval $(0,1)$ subject to the **initial conditions** $u(0) = \alpha$ and $u_x(0) = \beta$.

- For $\alpha = 1$, $\beta = 0$, and $f = 0$, the solution is $u(x) = 1$
- For $\alpha = 0$, $\beta = 1$, and $f = 0$, the solution is $u(x) = x$
- For $\alpha = 0$, $\beta = 0$, and $f(x) = \delta(x-t)$, let $u(x) = G(x,t)$ be the solution

Then in the general case we anticipate

$$u(x) = \alpha + \beta x + \int_0^1 G(x,t) f(t)\mathrm{d}t.$$

We have $G_{xx}(x,t) = 0$ **unless** $x = t$, so $G(x,t)$ is linear in x except possibly at $x = t$. This gives $G(x,t) = 0$ for $x \leq t$.
On the other hand, we have

$$1 = \int_{t-\varepsilon}^{t+\varepsilon} G_{xx}(x,t)\mathrm{d}x = G_x(t+\varepsilon) - G_x(t-\varepsilon) = G_x(t+\varepsilon),$$

so $G(x,t) = x - t$ for $x \geq t$.

Poisson equation in 1D

Consider now $u_{xx} = f$ on the interval $(0, 1)$ subject to the **boundary conditions** $u(0) = \alpha$ and $u(1) = \beta$.

- For $\alpha = 1$, $\beta = 0$, and $f = 0$, the solution is $u(x) = 1 - x$
- For $\alpha = 0$, $\beta = 1$, and $f = 0$, the solution is $u(x) = x$
- For $\alpha = 0$, $\beta = 0$, and $f(x) = \delta(x - t)$, let $u(x) = G(x, t)$ be the solution

So we anticipate

$$u(x) = \alpha(1 - x) + \beta x + \int_0^1 G(x, t) f(t) \mathrm{d}t.$$

$G(x, t)$ is linear in x except possibly at $x = t$. In the vicinity of $x = t$:

$$1 = \int_{t-\varepsilon}^{t+\varepsilon} G_{xx}(x, t) \mathrm{d}x = G_x(t + \varepsilon) - G_x(t - \varepsilon)$$

So $G(x, t) = kx$ for $x \leq t$, and $G(x, t) = (k + 1)x - t$ for $x \geq t$. From $G(1, t) = 0$ we get $k = t - 1$. The final result is

$$G(x, t) = \begin{cases} (t - 1)x & \text{for} \quad x \leq t \\ t(x - 1) & \text{for} \quad x \geq t \end{cases}$$