Lecture 29: Laplace eigenproblem on the disk and the Bessel functions

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Let us solve the eigenproblem

$$\Delta v = \lambda v \qquad \text{on} \quad \mathbb{D},$$

with the homogeneous Dirichlet boundary condition. Considering $v(r,\phi)$ for any fixed r as a function of ϕ, we can write

$$v(r,\phi) = \omega_0(r) + \sum_{n=1}^{\infty} \big(\omega_n(r)\cos n\phi + \omega_{-n}(r)\sin n\phi\big),$$

with $\omega_n(1) = 0$ for all $n \in \mathbb{Z}$. This leads to

$$(\omega_n)_{rr} + \frac{1}{r}(\omega_n)_r - \frac{n^2}{r^2}\omega_n = \lambda\omega_n.$$

With the substitutions $x = \alpha r$ and $y(x) = \omega_n(x/\alpha)$ where $\alpha = \sqrt{-\lambda}$, we get

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

This is called the **Bessel equation**.

Bessel equation

We can solve the Bessel equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0,$$

by power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Upon substitution, the equation reads

$$\sum_{k=0}^{\infty} k(k-1)a_k x^k + \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=2}^{\infty} a_{k-2} x^k - \sum_{k=0}^{\infty} n^2 a_k x^k = 0.$$

We have

- for $k = 0$: $n^2 a_0 = 0$, so $a_0 = 0$ unless $n = 0$
- for $k = 1$: $a_1 - n^2 a_1 = 0$, so $a_1 = 0$ unless $n = 1$
- for $k \geq 2$: $(k^2 - n^2)a_k + a_{k-2} = 0$

We have $a_k = -\dfrac{a_{k-2}}{k^2-n^2}$ for $k \geq 2$ and $k \neq n$. This gives $a_k = 0$ for all $k < n$, and for $k = n+1, n+3$, etc. The only possibly nonzero coefficients are a_n, a_{n+2}, a_{n+4}, etc. We calculate

$$a_{n+2q} = \frac{-a_{n+2(q-1)}}{2q \cdot 2(n+q)} = \frac{(-1)^q a_n}{2^q q! \cdot 2^q (n+q)(n+q-1)\cdots(n+1)} = \frac{(-1)^q n! a_n}{2^{2q} q! (n+q)!}.$$

Finally, a solution of the Bessel equation is

$$J_n(x) = \sum_{q=0}^{\infty} \frac{(-1)^q n! a_n}{2^{2q} q! (n+q)!} x^{n+2q} = n! 2^n a_n \sum_{q=0}^{\infty} \frac{(-1)^q}{q!(n+q)!} \left(\frac{x}{2}\right)^{n+2q},$$

where it is traditional to take $a_n = \frac{1}{n! 2^n}$. This is the *only* solution of the Bessel equation having a finite value at 0 (i.e., $|J_n(0)| < \infty$). The function J_n is called the **Bessel function** of the first kind, of order n. The convergence radius of the above series is ∞, so J_n is an *entire function*.

Properties of the Bessel functions

- $J_n(-x) = (-1)^n J_n(x)$
- $J_n(0) = \ldots = J_n^{(n-1)}(0) = 0$, and no positive zeroes of J_n are repeated
- $\left(x^n J_n(x)\right)' = x^n J_{n-1}(x)$, and $\left(x^{-n} J_n(x)\right)' = -x^{-n} J_{n+1}(x)$
- The zeroes of J_n and J_{n+1} are interlaced, and no two J_n and J_k have common zeroes
- $J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\pi n}{2}\right) + O\left(x^{-3/2}\right)$ for large x

J_0, J_1, and J_2