Lecture 28: Harmonic functions on the disk and the Poisson kernel

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Dirichlet-Laplace problem on the disk

Let us solve the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \mathbb{D}, \qquad u = g \quad \text{on} \quad \partial\mathbb{D},$$

where $\mathbb{D} = \{(x,y) : x^2 + y^2 < 1\}$ is the *unit disk*. In polar coordinates (r,ϕ), for any fixed $r \in (0,1)$, the function $u(r,\phi)$ is a function defined on a circle, so we can expand it into *Fourier series*

$$u(r,\phi) = \xi_0(r) + \sum_{n=1}^{\infty} \big(\xi_n(r)\cos n\phi + \xi_{-n}(r)\sin n\phi\big).$$

The boundary condition g is also a function of ϕ, so we can write

$$g(\phi) = \gamma_0 + \sum_{n=1}^{\infty} \big(\gamma_n \cos n\phi + \gamma_{-n}\sin n\phi\big).$$

We must require $\xi_n(1) = \gamma_n$ for all $n \in \mathbb{Z}$.

Substituting the expansion of u into the Laplace equation, we get

$$(\xi_n)_{rr} + \frac{1}{r}(\xi_n)_r - \frac{n^2}{r^2}\xi_n = 0,$$

whose only solutions that *do not blow up at* 0 are

$$\xi_n(r) = C_n r^n.$$

We find $C_n = \gamma_n$, and so

$$\begin{aligned} u(r,\phi) &= \gamma_0 + \sum_{n=1}^{\infty} r^n \big(\gamma_n \cos n\phi + \gamma_{-n} \sin n\phi\big) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\mathrm{d}\theta + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} \big(\cos n\theta \cos n\phi + \sin n\theta \sin n\phi\big) g(\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 + 2\sum_{n=1}^{\infty} r^n \cos n(\phi - \theta)\right) g(\theta)\mathrm{d}\theta. \end{aligned}$$

In particular,

$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\mathrm{d}\theta.$$

Poisson kernel

One can compute

$$1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1-r^2}{1+r^2-2r\cos x},$$

so

$$u(r,\phi) = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{g(\theta)\mathrm{d}\theta}{1+r^2-2r\cos(\phi-\theta)}.$$

With the **Poisson kernel**

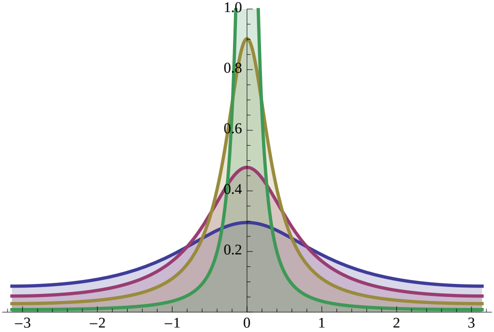
$$P_r(x) = \frac{1}{2\pi}\cdot\frac{1-r^2}{1+r^2-2r\cos x},$$

this can be written as

$$u(r,\phi) = \int_{-\pi}^{\pi} P_r(\phi-\theta)g(\theta)\mathrm{d}\theta = \int_{-\pi}^{\pi} g(\phi-\theta)P_r(\theta)\mathrm{d}\theta.$$

Properties of the Poisson kernel

- $P_r(x) = P_r(-x)$, and $P_r(x) \geq 0$ for all x and all $r \in (0,1)$
- $P_r(x) \to 0$ as $r \to 1$ for all $x \neq 0$
- $\int P_r(x) \mathrm{d}x = 1$



$r = 0.3, 0.5, 0.7, 0.9$