Lecture 25: Ring problems, Fourier series, and inhomogeneous boundary conditions

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Fourier series

Consider the Laplace eigenproblem on the unit circle S^1:

$$v'' = \lambda v.$$

The eigenfunctions are found to be

$$v_n(x) = \cos(nx), \qquad \text{with eigenvalue} \quad \lambda_n = -n^2, \qquad n = 0, 1, \ldots,$$

and

$$v_n(x) = \sin(nx), \qquad \text{with eigenvalue} \quad \lambda_n = -n^2, \qquad n = -1, -2, \ldots.$$

They are pairwise orthogonal:

$$\langle v_n, v_m \rangle = \int_0^{2\pi} v_n(x) v_m(x) \mathrm{d}x = (1 + \delta_{n0}) \pi \delta_{nm},$$

and they form a basis for the set of all functions f on the circle with $\|f\|^2 = \langle f, f \rangle < \infty$.

Fourier ring problem

Consider the heat equation on S^1

$$u_t = \Delta u, \qquad u(x,0) = f(x).$$

Suppose that u and f are written in terms of the **Fourier basis** $\{v_n\}$ as

$$u(x,t) = \sum_{n=-\infty}^{\infty} \xi_n(t) v_n(x), \qquad f = \sum_{n=-\infty}^{\infty} \beta_n v_n.$$

Then we have

$$u_t = \sum_{n=-\infty}^{\infty} (\xi_n)_t v_n, \quad \Delta u = \sum_{n=-\infty}^{\infty} \xi_n(-n^2) v_n \quad \Rightarrow \quad \xi_n(t) = \beta_n e^{-n^2 t},$$

so

$$u(x,t) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} \beta_n v_n(x).$$

Wave on a ring

Consider the wave equation on S^1:

$$u_{tt} = \Delta u, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Suppose

$$u(x,t) = \sum_{n=-\infty}^{\infty} \xi_n(t) v_n(x), \qquad f = \sum_{n=-\infty}^{\infty} \beta_n v_n, \qquad g = \sum_{n=-\infty}^{\infty} \gamma_n v_n.$$

Then we have

$$(\xi_n)_{tt} = -n^2 \xi_n, \qquad \Rightarrow \qquad \xi_n(t) = \beta_n \cos(nt) + \frac{\gamma_n}{n} \sin(nt),$$

so

$$u(x,t) = \beta_0 + \gamma_0 t + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\beta_n \cos(nt) + \frac{\gamma_n}{n} \sin(nt) \right) v_n(x).$$

Inhomogeneous Dirichlet boundary conditions

Consider the heat equation

$$u_t = \Delta u, \qquad u(x,0) = f(x), \qquad u(0,t) = \alpha, \qquad u(1,t) = \beta,$$

on $(0,1)$, where α, β are given numbers. For separation of variables, we need homogeneous boundary conditions (i.e., we need $\alpha = \beta = 0$). Let

$$\bar{u}(x) = \alpha + (\beta - \alpha)x,$$

so that $\bar{u}(0) = \alpha$ and $\bar{u}(1) = \beta$. This means that $w = u - \bar{u}$ should satisfy $w(0) = w(1) = 0$. We have

$$w_t = u_t, \qquad \Delta w = (u - \bar{u})_{xx} = u_{xx},$$

so by solving the problem

$$w_t = \Delta w, \qquad w(x,0) = f(x) - \bar{u}(x), \qquad w(0,t) = w(1,t) = 0,$$

with the *homogeneous* boundary conditions, and setting $u = \bar{u} + w$, we solve the original problem (with *inhomogeneous* boundary conditions).

Consider now the general problem

$$u_t = \Delta u, \qquad u(x,0) = f(x), \qquad u = g \quad \text{on} \quad \partial\Omega,$$

on $\Omega \subset \mathbb{R}^n$, where g does not depend on time. Suppose that $\bar{u}$ is some function defined on Ω, that is equal to g on the boundary $\partial\Omega$. Then $w = u - \bar{u}$ must satisfy $w = 0$ on $\partial\Omega$. We have

$$w_t = u_t, \qquad \Delta w = \Delta(u - \bar{u}) = \Delta u - \Delta\bar{u},$$

so the original problem reduces to

$$w_t = \Delta w + \Delta\bar{u}, \qquad w(x,0) = f(x) - \bar{u}(x), \qquad w = 0 \quad \text{on} \quad \partial\Omega.$$

This is a heat equation with an **inhomogeneous** (or a source) term. If we allow the boundary condition g to depend on time, we would get

$$w_t = \Delta w + \Delta\bar{u} - \bar{u}_t, \qquad w(x,0) = f(x) - \bar{u}(x,0), \qquad w = 0 \quad \text{on} \quad \partial\Omega.$$

Inhomogeneous Neumann boundary conditions

Now consider

$$u_t = \Delta u, \qquad u(x,0) = f(x), \qquad u_x(0,t) = \alpha, \qquad u_x(1,t) = \beta,$$

on $(0,1)$. Let

$$\bar{u}(x) = \alpha x + \frac{\beta - \alpha}{2} x^2,$$

so that $\bar{u}_x(0) = \alpha$ and $\bar{u}_x(1) = \beta$. We have

$$w_t = u_t, \qquad \Delta w = (u - \bar{u})_{xx} = u_{xx} - (\beta - \alpha),$$

so the original problem reduces to

$$w_t = \Delta w + \beta - \alpha, \qquad w(x,0) = f(x) - \bar{u}(x), \qquad w_x(0,t) = w_x(1,t) = 0.$$

Note that an inhomogeneous term arises again.

In general, inhomogeneous boundary conditions can be traded for inhomogeneous terms in the equation.

Inhomogeneous heat equation with a reaction term

So a problem of interest would be

$$u_t = \Delta u + \mu u + h, \qquad u(x,0) = f(x), \quad (x \in \Omega)$$

with some homogeneous boundary condition, where μ is a constant, and h is a function possibly depending on t. Let v_j and λ_j be eigenfunctions and eigenvalues of the Laplacian on Ω with the given boundary condition. Suppose

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t) v_j(x), \qquad h(x,t) = \sum_{j=1}^{\infty} \eta_j(t) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j.$$

Then the problem reduces to the ODE problems

$$(\xi_j)_t = (\lambda_j + \mu)\xi_j + \eta_j, \qquad \xi_j(0) = \beta_j.$$

The solution is

$$\xi_j(t) = e^{(\lambda_j+\mu)t}\beta_j + \int_0^t e^{(\lambda_j+\mu)(t-s)}\eta_j(s)\mathrm{d}s.$$