Lecture 24: Neumann boundary conditions and Fourier cosine series

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Math 319: Introduction to PDE
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Cosine series

Consider the Laplace eigenproblem on interval $(0,1)$ with the **homogeneous Neumann** boundary conditions:

$$v'' = \lambda v, \qquad v'(0) = v'(1) = 0.$$

The eigenfunctions and eigenvalues are found to be

$$v_k(x) = \cos(\pi k x), \qquad \lambda_k = -\pi^2 k^2, \qquad k = 0, 1, \ldots.$$

With respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) \mathrm{d}x,$$

the cosines $\{v_k\}$ are orthogonal:

$$\langle v_j, v_k \rangle = \int_0^1 \cos(\pi j x) \cos(\pi k x) \mathrm{d}x = \frac{1 + \delta_{k0}}{2} \delta_{jk},$$

and they form a basis for the set of all functions f with $\|f\|^2 = \langle f, f \rangle < \infty$.

Heat equation with Neumann boundary conditions

Consider the initial-boundary value problem on $(0,1)$

$$u_t = \Delta u, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad u(x,0) = f(x),$$

where $\Delta u = u_{xx}$. Suppose that u and f are written in terms of the **cosine basis** $\{v_k\}$ as

$$u(x,t) = \sum_{k=0}^{\infty} \xi_k(t) v_k(x), \qquad f = \sum_{k=0}^{\infty} \beta_k v_k.$$

Then we have

$$u_t = \sum_{k=0}^{\infty} (\xi_k)_t v_k, \quad \Delta u = \sum_{k=0}^{\infty} \xi_k \Delta v_k = \sum_{k=0}^{\infty} \xi_k(-\pi^2 k^2) v_k \quad \Rightarrow \quad \xi_k(t) = \beta_k e^{-\pi^2 k^2 t},$$

so

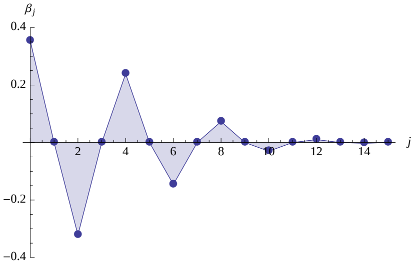
$$u(x,t) = \sum_{k=0}^{\infty} e^{-\pi^2 k^2 t} \beta_k \cos(\pi k x).$$

The essential difference to the Dirichlet case here is that there is a mode $(k=0)$ that **does not decay**.

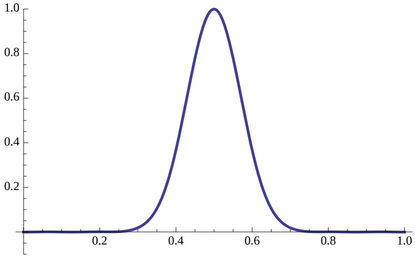
Example

Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x,0)$

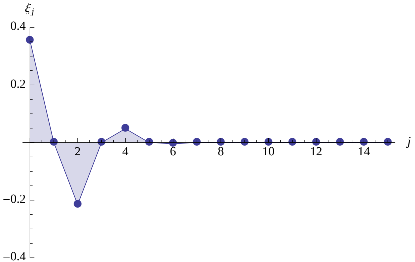


$u(x,0)$

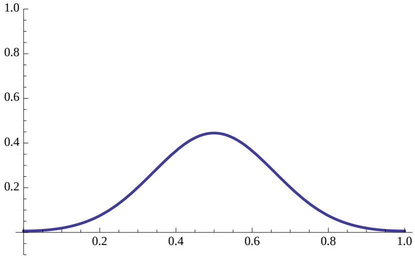


Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x, 0.1)$



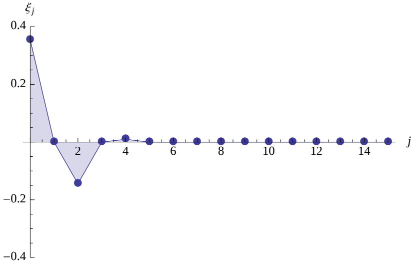
$u(x, 0.1)$



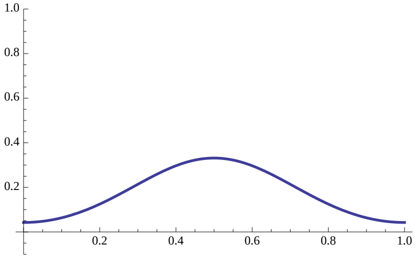
Example

Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x, 0.2)$



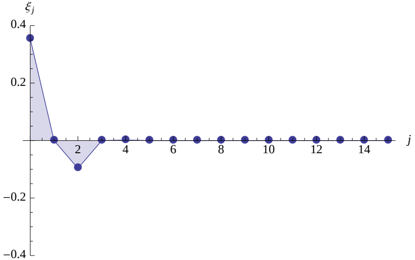
$u(x, 0.2)$



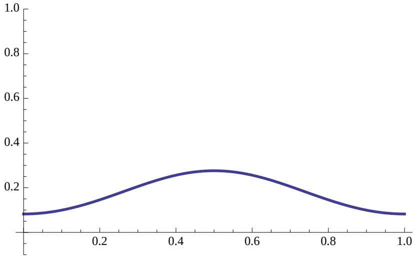
Example

Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x, 0.3)$

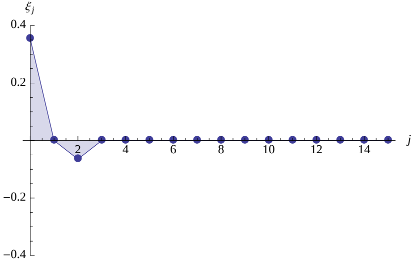


$u(x, 0.3)$

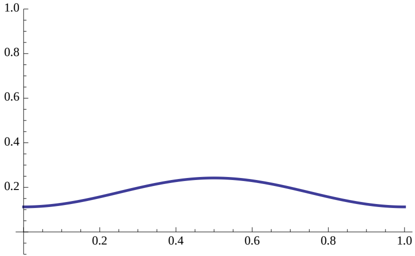


Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x, 0.4)$

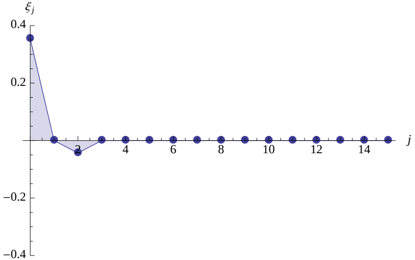


$u(x, 0.4)$

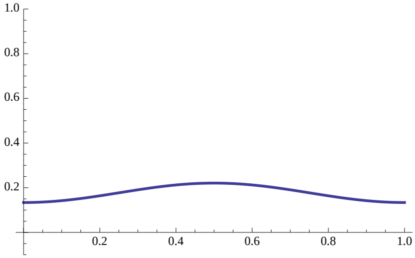


Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x, 0.5)$



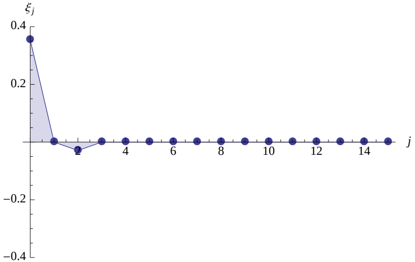
$u(x, 0.5)$



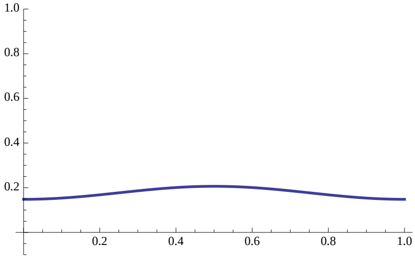
Example

Heat equation with initial condition $f(x) = e^{-100(x-1/2)^2}$.

Coefficients of $u(x, 0.6)$



$u(x, 0.6)$



Consider the initial-boundary value problem on $(0,1)$

$$u_{tt} = \Delta u, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Suppose

$$u(x,t) = \sum_{k=0}^{\infty} \xi_k(t) v_k(x), \qquad f = \sum_{k=0}^{\infty} \beta_k v_k, \qquad g = \sum_{k=0}^{\infty} \gamma_k v_k.$$

Then we have

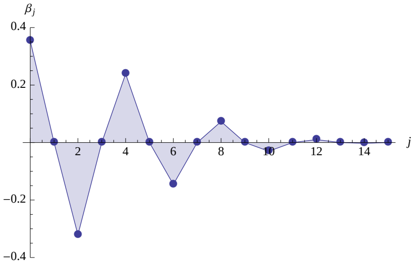
$$(\xi_k)_{tt} = -\pi^2 k^2 \xi_k, \qquad \Rightarrow \qquad \xi_k(t) = \beta_k \cos(\pi k t) + \frac{\gamma_k}{k} \sin(\pi k t),$$

so

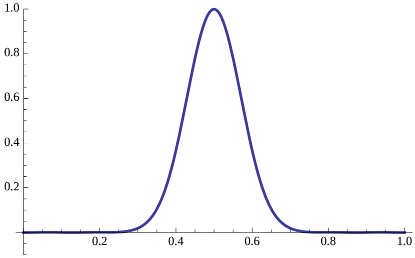
$$u(x,t) = \beta_0 + \gamma_0 t + \sum_{k=1}^{\infty} \Big(\beta_k \cos(\pi k t) + \frac{\gamma_k}{k} \sin(\pi k t)\Big) \cos(\pi k x).$$

Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x,0)$



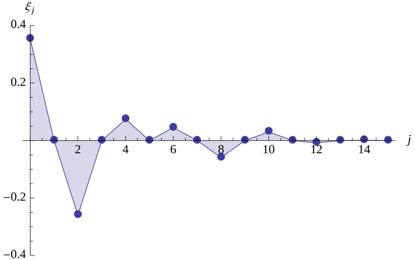
$u(x,0)$



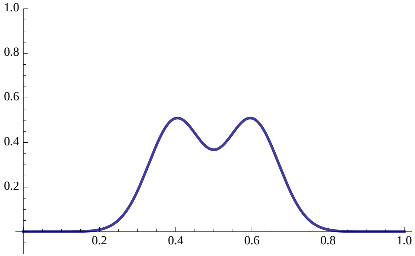
Example

Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.1)$

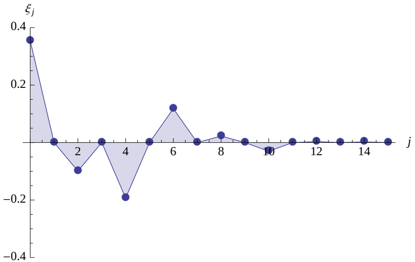


$u(x, 0.1)$

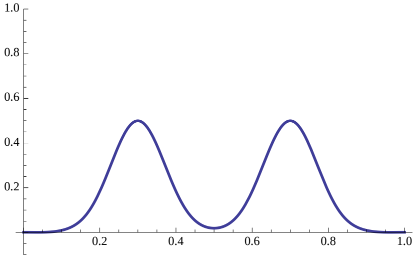


Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.2)$



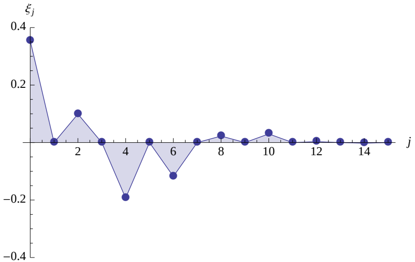
$u(x, 0.2)$



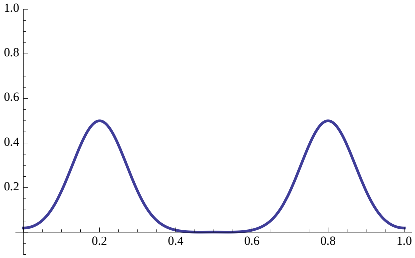
Example

Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.3)$

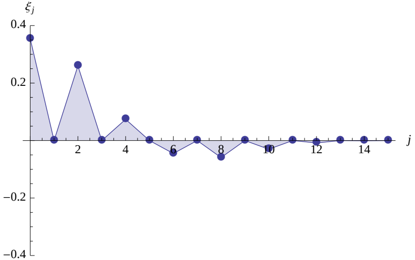


$u(x, 0.3)$

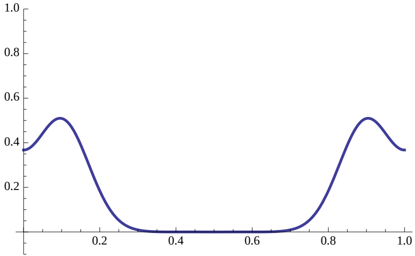


Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.4)$



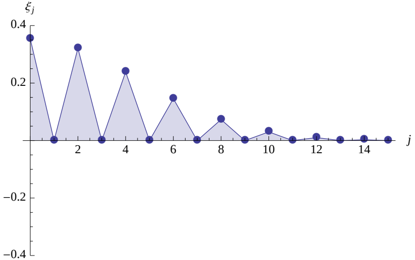
$u(x, 0.4)$



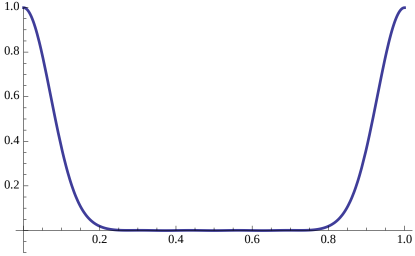
Example

Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.5)$



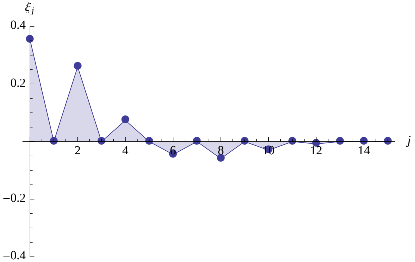
$u(x, 0.5)$



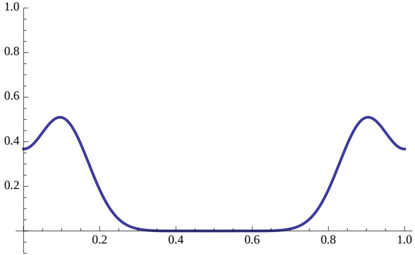
Example

Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.6)$

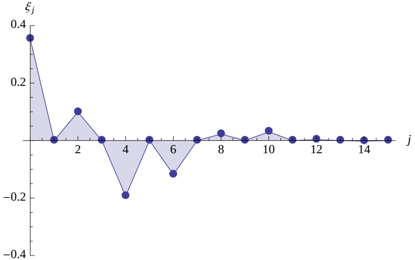


$u(x, 0.6)$

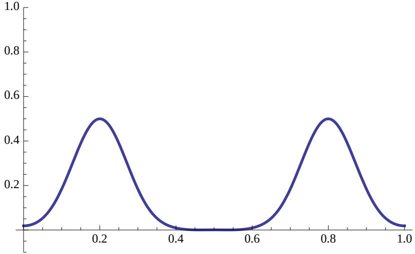


Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.7)$

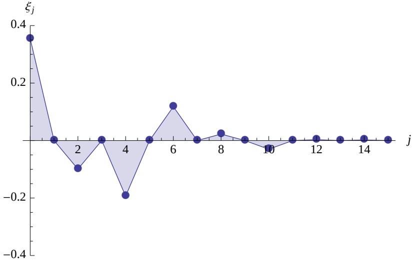


$u(x, 0.7)$



Wave equation with initial conditions $f(x) = e^{-100(x-1/2)^2}$ and $g \equiv 0$.

Coefficients of $u(x, 0.8)$



$u(x, 0.8)$