Lecture 23: Separation of variables for the heat and wave equations

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Recipe for the heat equation

Suppose Ω is a domain in $\mathbb{R}^n$, and consider the initial-boundary value problem

$$u_t = \Delta u, \qquad u(x,0) = f(x), \quad (x \in \Omega)$$

with some (homogeneous) boundary condition. Let v_j and λ_j be eigenfunctions and eigenvalues of the Laplacian on Ω with the given boundary condition

$$\Delta v_j = \lambda_j v_j, \qquad j = 1, 2, \ldots.$$

Suppose that u and f are written in terms of the eigenfunction basis $\{v_j\}$ as

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j.$$

Then we have

$$u(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} \beta_j v_j(x) = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle f, v_j \rangle v_j(x).$$

We are guaranteed that all $\lambda_j \leq 0$ and behave like $\lambda_j \sim -j^{2/n}$.

Recipe for the wave equation

Suppose Ω is a domain in $\mathbb{R}^n$, and consider the initial-boundary value problem

$$u_{tt} = \Delta u, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad (x \in \Omega)$$

with some (homogeneous) boundary condition. Suppose that

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j, \qquad g = \sum_{j=1}^{\infty} \gamma_j v_j.$$

Then we have

$$u(x,t) = \sum_{j=1}^{\infty} \left(\beta_j \cos\left(\sqrt{-\lambda_j} t\right) + \frac{\gamma_j}{\sqrt{-\lambda_j}} \sin\left(\sqrt{-\lambda_j} t\right) \right) v_j(x).$$

Recall that $\lambda_j \sim -j^{2/n}$, so $\sqrt{-\lambda_j} \sim j$ only when $n = 1$.

Boundary conditions

A boundary condition is a condition on the behavior of the solution u along the boundary $\partial\Omega$. For example, the **Robin boundary condition** is

$$\alpha u + \beta \partial_n u = \gamma,$$

where $\partial_n u$ is the normal derivative of u, and α, β, γ are given functions defined on the boundary.

- **Dirichlet** boundary condition is the case $\alpha = 1$ and $\beta = 0$
- **Neumann** boundary condition is the case $\alpha = 0$ and $\beta = 1$
- **Homogeneous** boundary condition means $\gamma = 0$
- It is possible to have different types of boundary conditions for different parts of the boundary
- Boundary conditions can be time dependent, but we will mainly consider time independent boundary conditions

For separation of variables, we need homogeneous boundary conditions. This is not a problem, because we can reduce nonhomogeneous boundary conditions into homogeneous ones.

Physical units and scaling

We studied the heat and wave equations (with hom. Dir. b.c.)

$$u_t - u_{xx} = 0, \qquad \text{and} \qquad u_{tt} - u_{xx} = 0,$$

resp., on the interval $(0, \pi)$. We derived, for instance, that the fundamental (i.e., lowest) frequency of the elastic string is $\frac{1}{2\pi}$. If we want to derive results in physical units, it suffices to do a simple scaling of the variables x and t.

In physical units, the above equations would be

$$u_\tau - k u_{\xi\xi} = 0, \qquad \text{and} \qquad u_{\tau\tau} - c^2 u_{\xi\xi} = 0,$$

on the interval $(0, L)$. It is easy to see that the scaling is

$$x = \frac{\pi}{L}\xi, \qquad t = \frac{\pi^2 k}{L^2}\tau, \qquad t = \frac{\pi c}{L}\tau,$$

with the latter two corresponding to the heat and wave, respectively.