Lecture 22: General discussions on PDE theory, ill-posedness

Gantumur Tsogtgerel

Math 319: Introduction to PDE
McGill University, Montréal

Tuesday March 1, 2011

What does solving a problem mean?

What is a problem? We need

- Set D, that represents all possible **data** in the problem
- Set S, that represents all possible **solutions**
- **Relation** $R(f,u) \in \{0,\dots\}$, defined for $f \in D$ and $u \in S$

Now the problem is: Given $f \in D$, find $u \in S$ such that $R(f,u) = 0$.

Example: $x^2 - a = 0$, with a as data, and x as the supposed solution.

- If we put $S = \mathbb{R}$ and $D = \mathbb{R}$, solution does not always exist
- If $S = \mathbb{C}$ and $D = \mathbb{C}$, there is always a solution
- In most cases we cannot compute the solution exactly
- But we have a very good idea about the solution(s)
- We can compute the solution approximately with any given accuracy

Problem solved: Add the gadget $\sqrt{a}$ to our bag of tools

Polynomial equations

Consider the equation

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0,$$

where $a_0, \ldots, a_n$ are data, and x is the solution.

- Case $n = 2$ is solved by $x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$ (2000BC ~ 16th century)
- $n = 3$ can be solved by using $\sqrt[3]{a}$ (Ferro, Tartaglia, Cardano ~1540)
- $n = 4$ can be solved by using $\sqrt[4]{a}$ (Ferrari ~1540)
- For $n \geq 5$ no general formula (Abel-Ruffini 1824, Galois 1832)
- Fundamental theorem: there exist n solutions (Argand 1806)
- Many analytic results, bounds etc.
- We can approximately compute the roots with any given accuracy

Satisfactorily solved.

Differentiation and integration

Differentiation: Given $f \in D$, find $u \in S$ such that $u - f' = 0$.

Take D and S to be the set of **elementary functions**, i.e., functions that can be formed using a finite combination of constants, arithmetic operations, radicals, exponential, logarithm, and composition.

> *The derivative of any given elementary function can be computed in a finite number of steps.*

Integration: Given $f \in D$, find $u \in S$ such that $u' - f = 0$.

> *There exist elementary functions whose integral is not elementary.*

Examples:

- $f(x) = e^{-x^2}$ (erf), $f(x) = 1/\log x$ (Li), $f(x) = \sin x/x$ (Si)
- $f(x) = 1/\sqrt{P(x)}$, where P is a polynomial of degree ≥ 3 with no repeated roots (elliptic integrals)

One can enrich $D = S$ by adding more functions to it, but there will always be lots of functions that cannot be integrated within the set.

Differential equations

- Obviously there is no hope of solving DEs in elementary terms
- Even accepting solutions involving integrals (like d'Alambert's and Poisson's formulas) does not help. There would be tons of DEs that cannot be solved.

So in general, we resort to **qualitative understanding**, complemented by the development of **good computational algorithms**. In retrospect, any method that addresses none of these two is of limited importance.

- Qualitative understanding can be gained from special solutions, numerical or physical experiments, and powerful analytic methods
- Qualitative understanding is necessary for developing and validating computational methods
- Good computational methods clearly add to qualitative understanding
- Representing the solution as rapidly converging series is very useful for both understanding and computation

Well- and ill-posedness

The problem $R(f,u)=0$ of finding $u \in S$ for given $f \in D$ is called **well-posed** if

- For any $f \in D$ there exists a unique solution $u \in S$,
- Varying f a bit results in a small variation of u.

  Meta-problem: Find "reasonable" sets D and S such that the problem $R(f,u)=0$ with $f \in D$ and $u \in S$ is well-posed, and that hopefully u can be computed efficiently.

In general, one has to extend the definition of R to bigger sets D and S than the original ones.

- Some problems are inherently **ill-posed**, i.e., not well-posed.
- Ill-posed problems should be replaced by well-posed ones if possible, but sometimes one is forced to "solve" ill-posed problems.

Backward heat equation is ill-posed

Consider the **backward heat** equation

$$u_t = -\Delta u, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x).$$

Suppose that u and f are written in terms of the sine basis $\{v_j\}$ as

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t)\, v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j.$$

Then we have

$$u_t = \sum_{j=1}^{\infty} (\xi_j)_t v_j, \qquad \Delta u = \sum_{j=1}^{\infty} \xi_j \Delta v_j = \sum_{j=1}^{\infty} \xi_j (-j^2)\, v_j \qquad \Rightarrow \qquad \xi_j(t) = \beta_j e^{+j^2 t},$$

so

$$u(x,t) = \sum_{j=1}^{\infty} e^{+j^2 t} \beta_j \sin(jx).$$

Note that the higher modes grow with unbounded rate.

Elliptic initial value problems are ill-posed

Consider the **Laplace initial value problem**

$$u_{xx} + u_{tt} = 0, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Suppose

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j, \qquad g = \sum_{j=1}^{\infty} \gamma_j v_j.$$

Then we have

$$(\xi_j)_{tt} - j^2 \xi_j = 0, \qquad \Rightarrow \qquad \xi_j(y) = \beta_j \cosh(jy) + \frac{\gamma_j}{j} \sinh(jy).$$

Note that the higher modes grow with unbounded rate.

Also, **inverse problems** are usually ill-posed.