Lecture 21: Introduction to separation of variables

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Math 319: Introduction to PDE

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Poisson problem on interval

We considered the following problem on the interval $(0, \pi)$

$$u_{xx} = f, \qquad u(0) = u(\pi) = 0.$$

We view this as inverting the operator $\Delta : v \mapsto v_{xx}$. We found the **eigenfunctions** and **eigenvalues** of Δ to be

$$v_j(x) = \sin(jx), \qquad \text{and} \qquad \lambda_j = -j^2, \qquad \big(j = 1, 2, \ldots\big).$$

We take the followings facts as given.

- Any function f with $\|f\| < \infty$ satisfies $f = \sum_{j=1}^{\infty} \beta_j v_j$ in the L^2-sense, with the unique coefficients $\beta_j = \frac{2}{\pi}\langle f, v_j \rangle$.
- If $u = \sum_{j=1}^{\infty} \xi_j v_j$ in L^2, then $u_{xx} = \left(\sum_{j=1}^{\infty} \xi_j v_j\right)_{xx} = \sum_{j=1}^{\infty} \xi_j \left(v_j\right)_{xx}$.

From those we immediately get

$$u_{xx} = \sum_{j=1}^{\infty} (-j^2) \xi_j v_j, \qquad \text{and so} \qquad u = \sum_{j=1}^{\infty} \frac{1}{(-j^2)} \beta_j v_j.$$

Initial value problem for ODE

Consider the problem of finding $x(t) \in \mathbb{R}^n$ satisfying

$$x_t = Ax, \qquad x(0) = b,$$

where A is an $n \times n$ symmetric matrix, and $b \in \mathbb{R}^n$ is a given initial state. There exist orthonormal set of eigenvectors $v_1, \ldots, v_n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$Av_i = \lambda_i v_i, \qquad \text{with} \qquad \langle v_i, v_k \rangle \equiv v_i^T v_k = \delta_{ik}.$$

Suppose that x and b are written in terms of the basis $\{v_i\}$ as

$$x(t) = \sum_i \xi_i(t) v_i, \qquad b = \sum_i \beta_i v_i.$$

Then we have

$$x_t = \sum_i (\xi_i)_t v_i, \qquad Ax = \sum_i \xi_i A v_i = \sum_i \xi_i \lambda_i v_i \qquad \Rightarrow \qquad \xi_i(t) = \beta_i e^{\lambda_i t},$$

so

$$x(t) = \sum_i \beta_i e^{\lambda_i t} v_i = \sum_i e^{\lambda_i t} \langle b, v_i \rangle v_i.$$

Heat equation on interval

Consider the initial-boundary value problem on $(0, \pi)$

$$u_t = \Delta u, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x),$$

where $\Delta u = u_{xx}$. Suppose that u and f are written in terms of the sine basis $\{v_j\}$ as

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j.$$

Then we have

$$u_t = \sum_{j=1}^{\infty} (\xi_j)_t v_j, \qquad \Delta u = \sum_{j=1}^{\infty} \xi_j \Delta v_j = \sum_{j=1}^{\infty} \xi_j (-j^2) v_j \qquad \Rightarrow \qquad \xi_j(t) = \beta_j e^{-j^2 t},$$

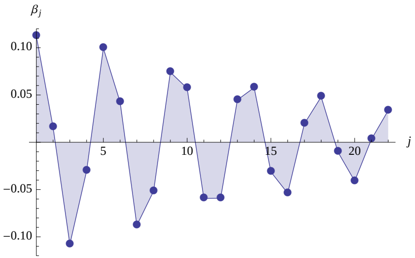
so

$$u(x,t) = \sum_{j=1}^{\infty} e^{-j^2 t} \beta_j \sin(jx).$$

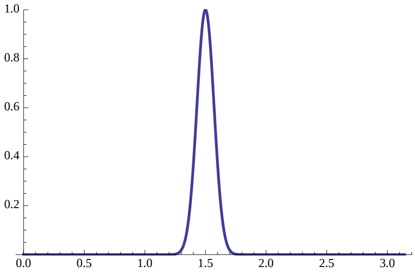
Example

Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x,0)$

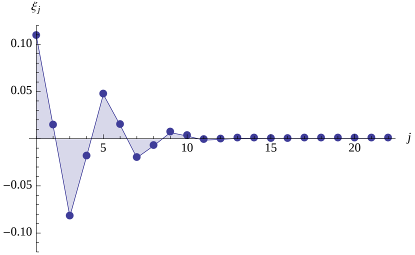


$u(x,0)$

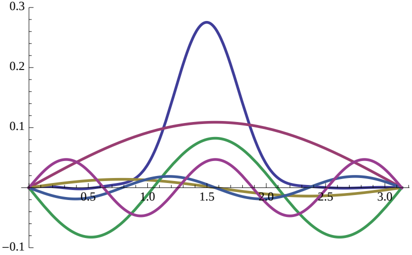


Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x, 0.03)$

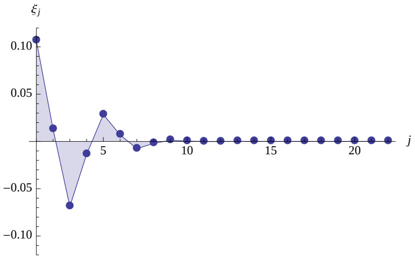


Frequency components of $u(x, 0.03)$.
Blue curve is $u(x, 0.03)$.

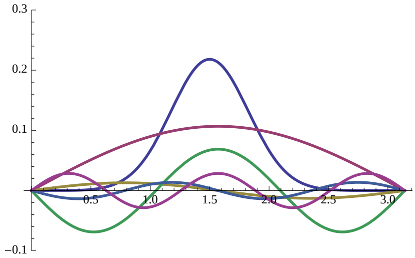


Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x, 0.05)$

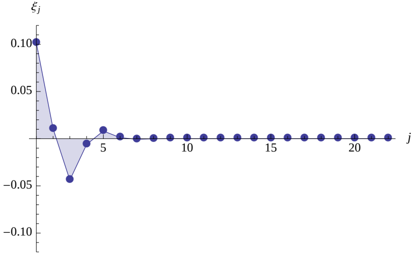


Frequency components of $u(x, 0.05)$.
Blue curve is $u(x, 0.05)$.

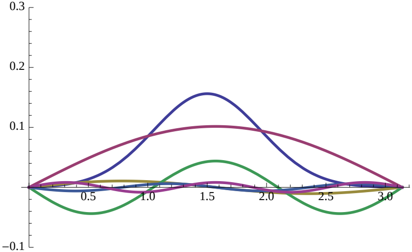


Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x, 0.1)$



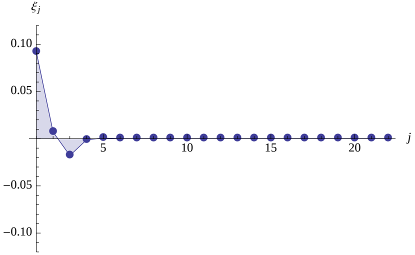
Frequency components of $u(x, 0.1)$.
Blue curve is $u(x, 0.1)$.



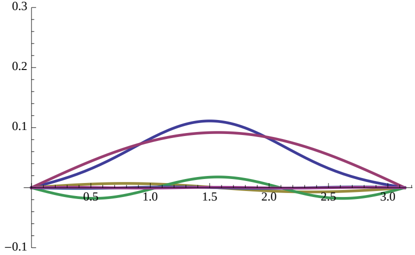
Example

Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x, 0.2)$

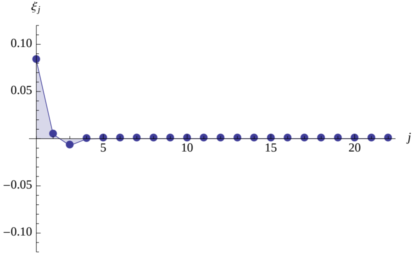


Frequency components of $u(x, 0.2)$.
Blue curve is $u(x, 0.2)$.

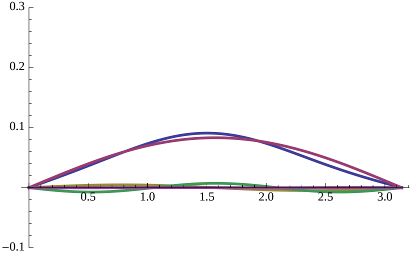


Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x, 0.3)$

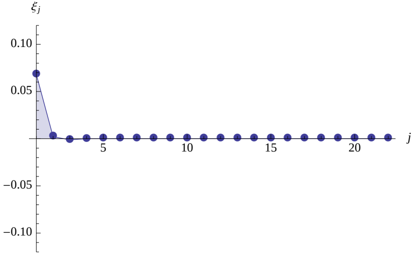


Frequency components of $u(x, 0.3)$.
Blue curve is $u(x, 0.3)$.

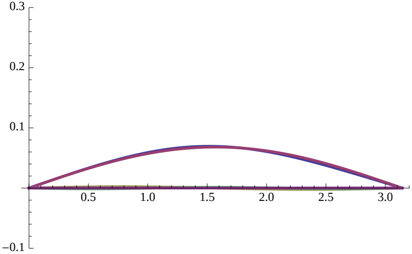


Heat equation with initial condition $f(x) = e^{-200(x-3/2)^2}$.

Coefficients of $u(x, 0.5)$



Frequency components of $u(x, 0.5)$.
Blue curve is $u(x, 0.5)$.



Alternative derivation

Consider the problem

$$u_t = u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x),$$

on $(0,\pi)$. Suppose that we can write $u(x,t) = X(x)T(t)$, i.e., *variables separate*. Then we have

$$u_t(x,t) = X(x)T'(t), \qquad u_{xx}(x,t) = X''(x)T(t) \quad \Rightarrow \quad X(x)T'(t) = X''(x)T(t).$$

Dividing through by $X(x)T(t)$, we get

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

The left hand side depends only on t, while the right hand side depends only on x. Hence the both sides must equal to a constant:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{const.}$$

This gives the two equations

$$T'(t) = \lambda T(t), \qquad X''(x) = \lambda X(x).$$

The second equation with the boundary conditions $X(0) = X(\pi) = 0$, has the solution

$$X_j(x) = \sin(jx), \qquad \lambda_j = -j^2,$$

for every positive integer j. Then the first equation is solved by

$$T_j(t) = T_j(0)e^{-j^2 t}.$$

By forming a linear combination of infinitely many solutions $u_j(x,t) = e^{-j^2 t}\sin(jx)$, we get

$$u(x,t) = \sum_{j=1}^{\infty} C_j e^{-j^2 t} \sin(jx).$$

So we are back to the questions related to the eigenfunctions of Δ.

Consider the initial-boundary value problem on $(0,\pi)$

$$u_{tt} = \Delta u, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Suppose

$$u(x,t) = \sum_{j=1}^{\infty} \xi_j(t) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j, \qquad g = \sum_{j=1}^{\infty} \gamma_j v_j.$$

Then we have

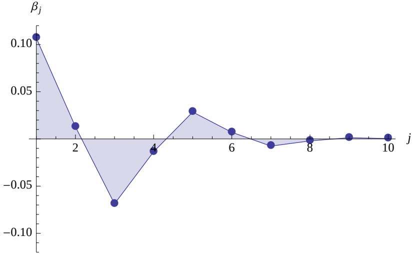
$$(\xi_j)_{tt} = -j^2 \xi_j, \qquad \Rightarrow \qquad \xi_j(t) = \beta_j \cos(jt) + \frac{\gamma_j}{j} \sin(jt),$$

so

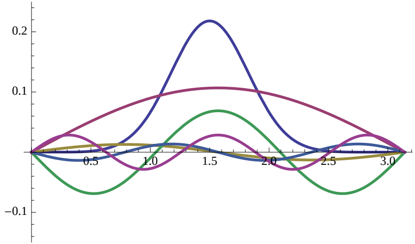
$$u(x,t) = \sum_{j=1}^{\infty} \left(\beta_j \cos(jt) + \frac{\gamma_j}{j} \sin(jt) \right) \sin(jx).$$

Wave equation example with $g \equiv 0$

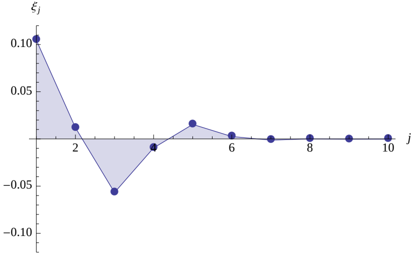
Coefficients of $u(x,0)$



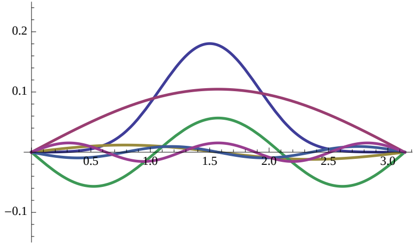
Frequency components of $u(x,0)$.
Blue curve is $u(x,0)$.



Coefficients of $u(x, 0.2)$

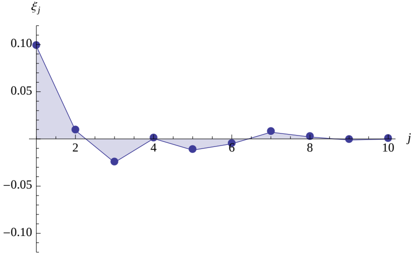


Frequency components of $u(x, 0.2)$.
Blue curve is $u(x, 0.2)$.

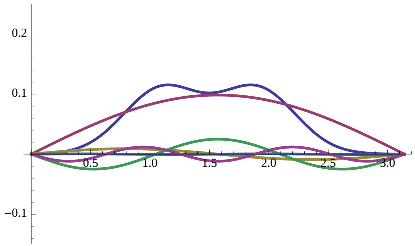


Wave equation example with $g \equiv 0$

Coefficients of $u(x, 0.4)$

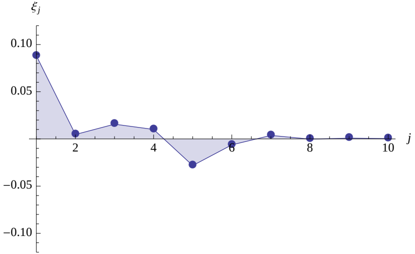


Frequency components of $u(x, 0.4)$.
Blue curve is $u(x, 0.4)$.

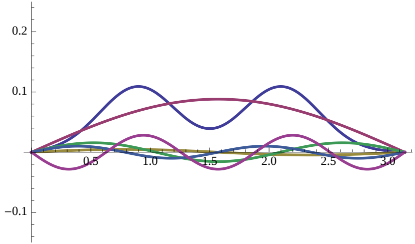


Wave equation example with $g \equiv 0$

Coefficients of $u(x, 0.6)$

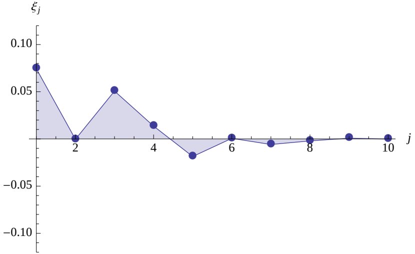


Frequency components of $u(x, 0.6)$.
Blue curve is $u(x, 0.6)$.

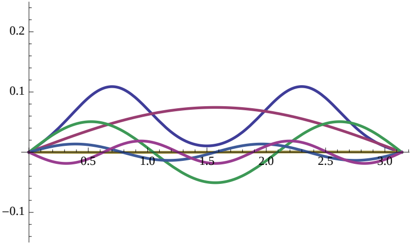


Wave equation example with $g \equiv 0$

Coefficients of $u(x, 0.8)$

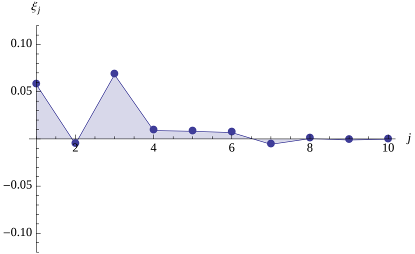


Frequency components of $u(x, 0.8)$.
Blue curve is $u(x, 0.8)$.

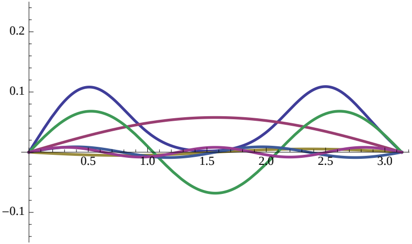


Wave equation example with $g \equiv 0$

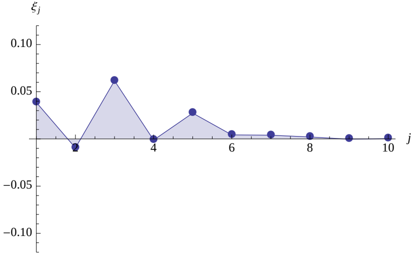
Coefficients of $u(x, 1.0)$



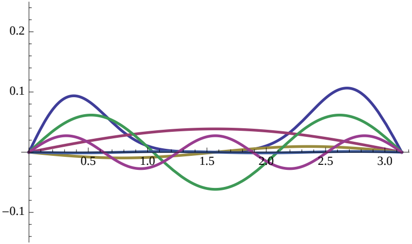
Frequency components of $u(x, 1.0)$.
Blue curve is $u(x, 1.0)$.



Coefficients of $u(x, 1.2)$

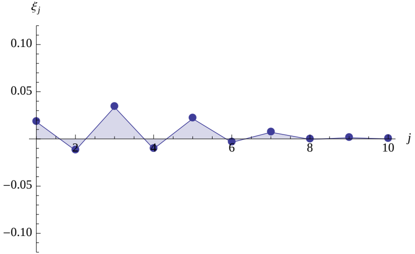


Frequency components of $u(x, 1.2)$.
Blue curve is $u(x, 1.2)$.

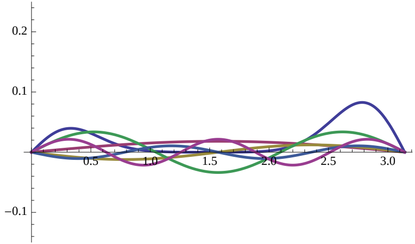


Wave equation example with $g \equiv 0$

Coefficients of $u(x, 1.4)$

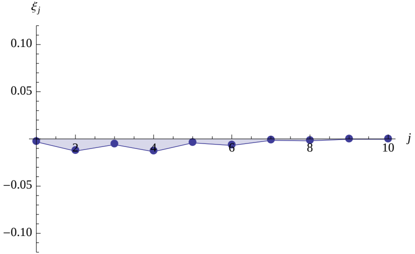


Frequency components of $u(x, 1.4)$.
Blue curve is $u(x, 1.4)$.

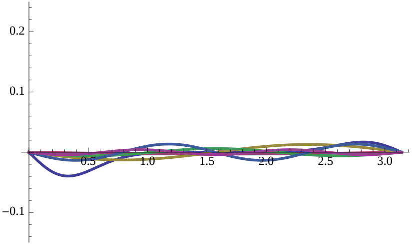


Wave equation example with $g \equiv 0$

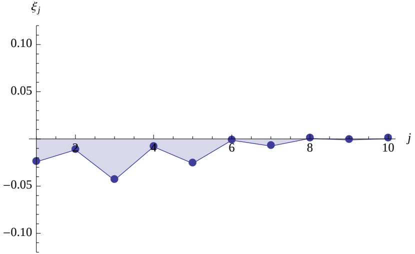
Coefficients of $u(x, 1.6)$



Frequency components of $u(x, 1.6)$.
Blue curve is $u(x, 1.6)$.

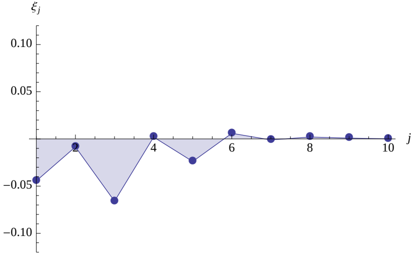


Coefficients of $u(x, 1.8)$

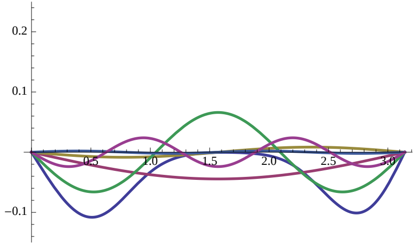


Frequency components of $u(x, 1.8)$.
Blue curve is $u(x, 1.8)$.

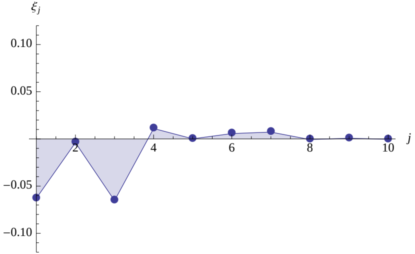
Coefficients of $u(x, 2.0)$



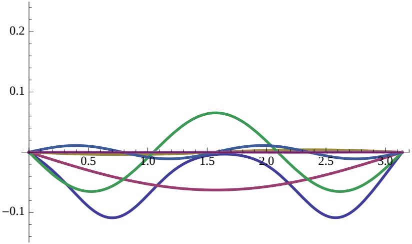
Frequency components of $u(x, 2.0)$.
Blue curve is $u(x, 2.0)$.



Coefficients of $u(x, 2.2)$



Frequency components of $u(x, 2.2)$.
Blue curve is $u(x, 2.2)$.



Consider the boundary value problem

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(\pi,y) = 0, \qquad u(x,0) = f(x), \quad u(x,a) = g(x),$$

on $(0,\pi) \times (0,a)$. Suppose

$$u(x,y) = \sum_{j=1}^{\infty} \xi_j(y) v_j(x), \qquad f = \sum_{j=1}^{\infty} \beta_j v_j, \qquad g = \sum_{j=1}^{\infty} \gamma_j v_j.$$

Then we have

$$(\xi_j)_{yy} - j^2 \xi_j = 0, \qquad \Rightarrow \qquad \xi_j(y) = \beta_j \cosh(jy) + \delta_j \sinh(jy),$$

where $\delta_j = \frac{\gamma_j - \beta_j \cosh(ja)}{\sinh(ja)}$. The final solution is

$$u(x,y) = \sum_{j=1}^{\infty} \big(\beta_j \cosh(jy) + \delta_j \sinh(jy)\big) \sin(jx).$$