Lecture 20: Fourier sine series (continued)

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Math 319: Introduction to PDE
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Poisson problem on interval

We are considering the following problem on the interval $(0, \pi)$

$$u_{xx} = f, \qquad u(0) = u(\pi) = 0.$$

We view this as inverting the operator $\Delta : v \mapsto v_{xx}$. We found the **eigenfunctions** and **eigenvalues** of Δ to be

$$v_j(x) = \sin(jx), \qquad \text{and} \qquad \lambda_j = -j^2, \qquad \big(j = 1, 2, \ldots\big).$$

We take the followings facts as given.

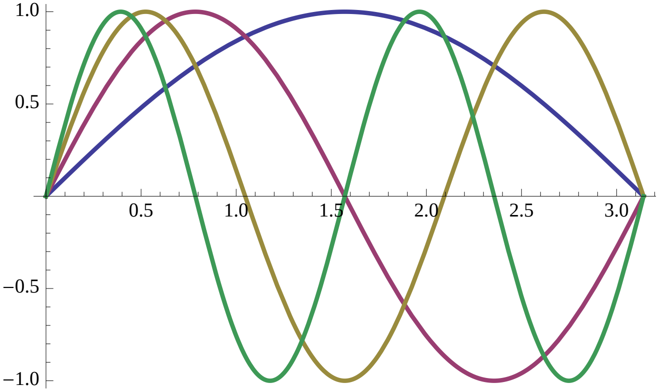
- Any function f with $\|f\| < \infty$ satisfies $f = \sum_{j=1}^{\infty} \beta_j v_j$ in the L^2-sense, with the unique coefficients $\beta_j = \frac{2}{\pi}\langle f, v_j \rangle$.
- If $u = \sum_{j=1}^{\infty} \xi_j v_j$ in L^2, then $u_{xx} = \left(\sum_{j=1}^{\infty} \xi_j v_j\right)_{xx} = \sum_{j=1}^{\infty} \xi_j \left(v_j\right)_{xx}$.

From those we immediately get

$$u_{xx} = \sum_{j=1}^{\infty} (-j^2) \xi_j v_j, \qquad \text{and so} \qquad u = \sum_{j=1}^{\infty} \frac{1}{(-j^2)} \beta_j v_j.$$

Fourier sine basis

$v_1(x) = \sin x,\ v_2(x) = \sin 2x,\ v_3(x) = \sin 3x,$ and $v_4(x) = \sin 4x.$

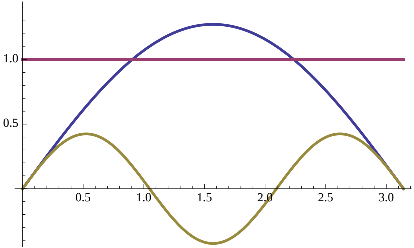


Example

Solving $u_{xx} = 1$ on $(0, \pi)$ with the boundary conditions $u(0) = u(\pi) = 0$. The solution is $u(x) = \frac{1}{2}x(x - \pi)$.

$$1 = \sum_{j=1}^{\infty} \frac{2(1-(-1)^j)}{j\pi} \sin jx.$$

1 term with next update:



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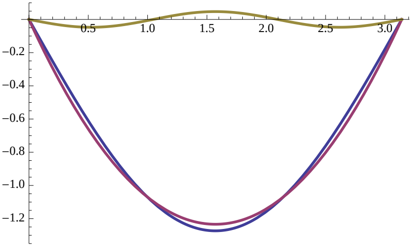

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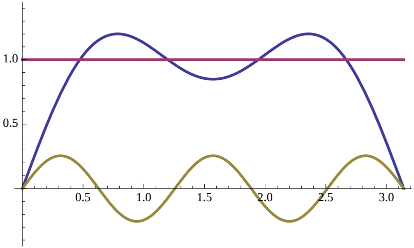
3 terms with next update:



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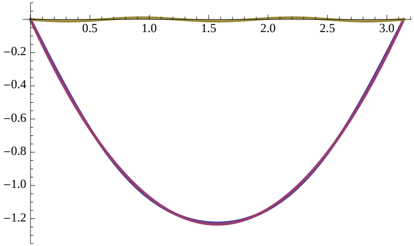

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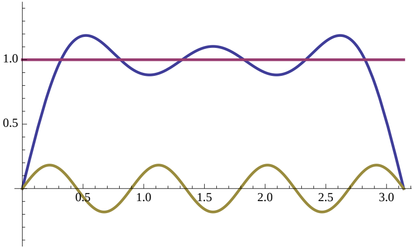
5 terms with next update:



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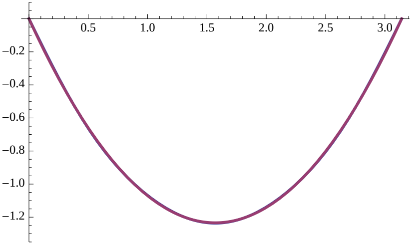

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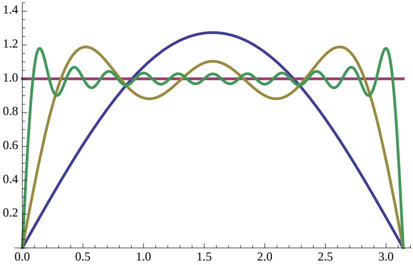
1, 5, 21 terms:



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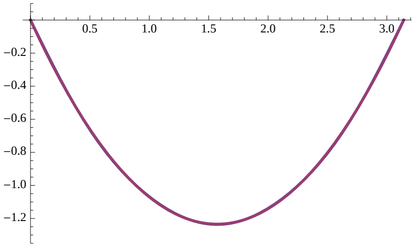

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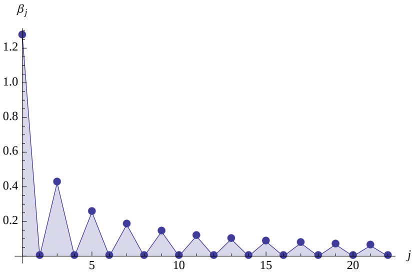

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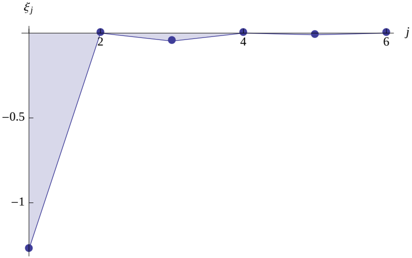

Orthogonality

Let g be a function with $\|g\| < \infty$, and let

$$g_n = \sum_{j=1}^{n} \langle g, v_j \rangle v_j,$$

be its **truncated sine series** (We normalize v_j so that $\|v_j\| = 1$). Then

$$\langle g - g_n, v_k \rangle = \langle g, v_k \rangle - \left\langle \sum_{j=1}^{n} \langle g, v_j \rangle v_j, v_k \right\rangle = \langle g, v_k \rangle - \sum_{j=1}^{n} \langle g, v_j \rangle \langle v_j, v_k \rangle = 0,$$

if $k \leq n$. This means $g - g_n$ is **orthogonal** to the subspace spanned by $v_1, \ldots, v_n$. In particular, $g - g_n \perp g_n$. We have the Pythagorean theorem:

$$\begin{aligned} \|g\|^2 = \langle g, g \rangle &= \langle g - g_n + g_n, g - g_n + g_n \rangle \\ &= \langle g - g_n, g - g_n \rangle + \langle g - g_n, g_n \rangle + \langle g_n, g - g_n \rangle + \langle g_n, g_n \rangle \\ &= \langle g - g_n, g - g_n \rangle + \langle g_n, g_n \rangle = \|g - g_n\|^2 + \|g_n\|^2. \end{aligned}$$

This implies **Bessel's inequality**:

$$\|g_n\| \leq \|g\|.$$

Optimality of the truncated series

Let

$$S_n = \operatorname{span}\{v_1, \ldots, v_n\},$$

and let $h \in S_n$ be an arbitrary element of S_n. Then $w = g_n - h \in S_n$, and

$$\begin{aligned}
\|g-h\|^2 &= \langle g - g_n + g_n - h, g - g_n + g_n - h\rangle = \langle g - g_n + w, g - g_n + w\rangle \\
&= \langle g - g_n, g - g_n\rangle + \langle g - g_n, w\rangle + \langle w, g - g_n\rangle + \langle w, w\rangle \\
&= \langle g - g_n, g - g_n\rangle + \langle w, w\rangle = \|g - g_n\|^2 + \|w\|^2.
\end{aligned}$$

This means

$$\|g - g_n\| \leq \|g - h\| \qquad \text{for any} \quad h \in S_n,$$

or equivalently

$$\|g - g_n\| = \min_{h \in S_n} \|g - h\|.$$

So $g_n = \sum_{j=1}^{n} \langle g, v_j\rangle v_j$ is as close to g as one gets by using linear combinations of the first n eigenfunctions $v_1, \ldots, v_n$.

General considerations:

- Classification by order, by linear or nonlinear
- Change of variables in PDE
- Classification of linear second order PDEs
- Canonical forms

Model elliptic equations are **Laplace** and **Poisson**:

- Derivation of the equations in electrostatics
- Interpretation as steady state of heat or wave equations
- Dirichlet principle, Dirichlet energy, uniqueness
- 5-point discretization, discrete maximum principle
- Method of electrostatic images

Model hyperbolic equations are **wave** and **advection**:

- Characteristic coordinates, method of characteristics
- Simple discretization of advection equation, CFL condition
- Derivation of wave equation for elastic string
- D'Alambert's solution of the 1D wave equation
- Poisson's formula for 3D waves, and method of descent for 2D waves
- Energy inequality, uniqueness

Model parabolic equation is **heat**:

- Derivation from heat conduction problem
- Self-similar solutions, fundamental solution
- Discretization by explicit and implicit methods, discrete maximum principle