Lecture 2: Linear algebra and simple ODEs

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The simplest differential equation

Find $u:\mathbb{R}\to\mathbb{R}$ such that

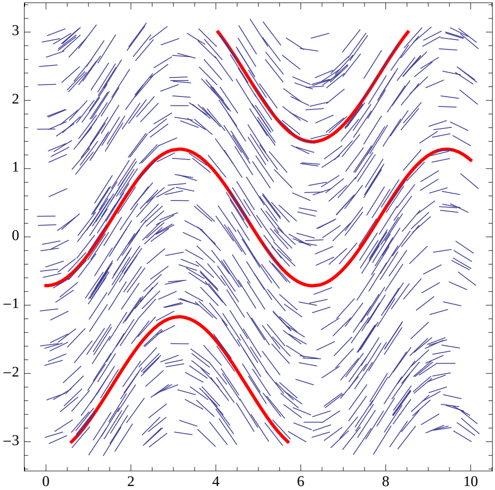
$$u_x = f,$$

with e.g. $f(x) = \sin x$.

Direct integration gives

$$u(x) - u(0) = \int_0^x f(t)\,\mathrm{d}t.$$

Integration generalizes to solving differential equations.

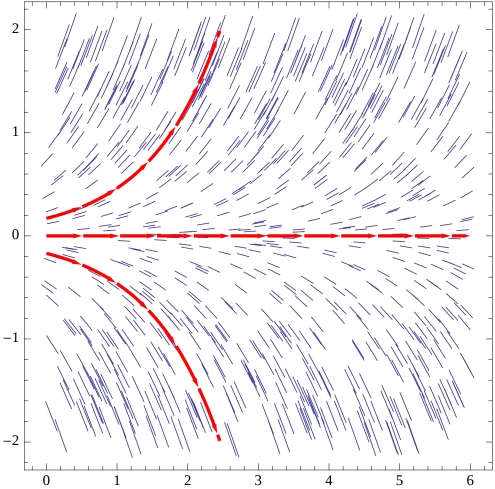


Second example

$u_x = u.$

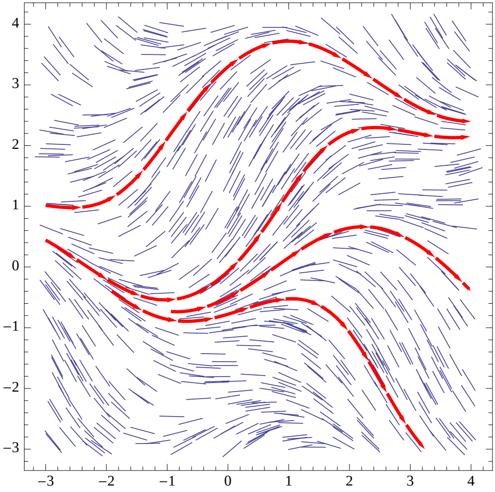
The solutions are

$u(x) = u(0)\,e^x.$



Third example

$$u_x = \sin u + \cos x.$$



Linearity

Let us consider the operation of forming the function u_x out of the function u:

$$Du := u_x.$$

We call D an **operator** acting on functions.

- For any $\alpha \in \mathbb{R}$, and any **differentiable** function u, we have

$$D(\alpha u) = \alpha \cdot Du.$$

- For any functions u and v both **differentiable**, we have

$$D(u + v) = Du + Dv.$$

If D is an operator that satisfies the preceding two conditions, then we say D is a **linear operator**.

The differential equation $u_x = f$ can be rewritten as

$$Du = f.$$

This looks an awful lot like the linear algebraic equation

$$Ay = b.$$

The similarity is deeper, because D and A are both **linear**. The only difference is that while A is a matrix acting on *n-dimensional vectors*, D is an operator acting on *functions*.

For concreteness, let $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$, and let $C^1(\mathbb{R})$ be the space of continuously differentiable functions. Then

$$D : C^1(\mathbb{R}) \to C(\mathbb{R}), \qquad \text{and} \qquad A : \mathbb{R}^n \to \mathbb{R}^m.$$

Trying to solve $Du = f$ means essentially trying to invert D.

The **range** (or column space) and the **kernel** (or null space) of A are

$$\operatorname{Ran} A = \{Ay : y \in \mathbb{R}^n\}, \qquad \text{and} \qquad \operatorname{Ker} A = \{y : Ay = 0\}.$$

- If $b \in \operatorname{Ran} A$ then $Ay = b$ has a solution (**existence**).
- Suppose that $Ay = b$ and $Az = b$. Then

  $$A(y-z) = Ay - Az = b - b = 0,$$

  so $y - z \in \operatorname{Ker} A$.
- If $\operatorname{Ker} A = \{0\}$, then $y = z$ (**uniqueness**).
- If $Ay = b$ and $q \in \operatorname{Ker} A$, then

  $$A(y+q) = Ay + Aq = b + 0 = b,$$

  so $y + q$ is also a solution.
- If $Ay = b$ then $\{y + q : q \in \operatorname{Ker} A\}$ is the **set of all solutions**.

We have $u_x(x) \approx \dfrac{u(x+h)-u(x)}{h}$ for small h, if e.g. $u \in C^1(\mathbb{R})$. So $u_x = f$ is something like

$$y_{i+1} - y_i = b_i,$$

where $y_i \approx u(ih)$ and $b_i \approx hf(ih)$. This can be solved as

$$y_n = y_{n-1} + b_{n-1} = y_{n-2} + b_{n-2} + b_{n-1} = \ldots = y_0 + b_0 + \ldots + b_{n-1}.$$

Let us rewrite the equation as

$$Ay = b,$$

with

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{n\times(n+1)}.$$

The dimension of $\operatorname{Ker} A$ is 1.