Lecture 19: Fourier sine series

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Solving linear systems using eigenvector bases

With A a symmetric $n \times n$ matrix, and $b \in \mathbb{R}^n$, we consider the linear equation

$$Ax = b.$$

There exist orthonormal set of eigenvectors $v_1, \dots, v_n$, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$:

$$Av_i = \lambda_i v_i, \qquad \text{with} \qquad \langle v_i, v_k \rangle \equiv v_i^T v_k = \delta_{ik}.$$

Suppose that x and b are written in terms of the basis $\{v_i\}$ as

$$x = \sum_i \xi_i v_i, \qquad b = \sum_i \beta_i v_i.$$

Then we have

$$Ax = \sum_i \xi_i A v_i = \sum_i \xi_i \lambda_i v_i = \sum_i \beta_i v_i, \qquad \Rightarrow \qquad \xi_i = \beta_i / \lambda_i.$$

The coordinates β_i can be found by

$$\langle v_k, b \rangle = \sum_i \beta_i \langle v_k, v_i \rangle = \sum_i \beta_i \delta_{ki} = \beta_k.$$

Poisson problem on interval

Now consider the following boundary value problem on the interval $(0,\pi)$

$$u_{xx} = f, \qquad u(0) = u(\pi) = 0.$$

We view this as inverting the operator $\Delta : u \mapsto u_{xx}$, acting on twice differentiable functions that vanish at 0 and 1. Let us try to find the **eigenvalues** and **eigenfunctions** of Δ. The problem to be solved is

$$v_{xx} = \lambda v, \qquad v(0) = v(\pi) = 0.$$

The solutions are

$$v_j(x) = \sin(jx), \qquad j = 1, 2, \ldots,$$

with the eigenvalues $\lambda_j = -j^2$. Supposing that

$$u = \sum_{j=1}^{\infty} \xi_j v_j, \qquad f = \sum_{j=1}^{\infty} \beta_j v_j,$$

we have

$$\Delta u = \sum_{j=1}^{\infty} \xi_j \Delta v_j = \sum_{j=1}^{\infty} \xi_j \lambda_j v_j = \sum_{j=1}^{\infty} \beta_j v_j.$$

Poisson problem on interval

As in the matrix-vector case, we want to conclude that $\xi_j = \beta_j / \lambda_j$. Several questions arise:

- Can we really write any given function f as $f = \sum_{j=1}^{\infty} \beta_j v_j$ with some coefficients β_j?
- What is the sense in which f is equal to the infinite sum $\sum_{j=1}^{\infty} \beta_j v_j$?
- How do we compute β_j? Are these unique?
- Can we conclude from $\sum_{j=1}^{\infty} \xi_j \lambda_j v_j = \sum_{j=1}^{\infty} \beta_j v_j$ that $\xi_j \lambda_j = \beta_j$?
- Is it true that $\Delta \sum_{j=1}^{\infty} \xi_j v_j = \sum_{j=1}^{\infty} \xi_j \Delta v_j$?

In this course, we can only give partial answers to these questions. Let us start with Question 3. Define the L^2-**inner product**

$$\langle v, w \rangle = \int_0^{\pi} v(x) w(x) \mathrm{d}x.$$

Inner product

We calculate

$$\langle v_j, v_k\rangle = \int_0^\pi \sin(jx)\sin(kx)\mathrm{d}x = \int_0^\pi \frac{\cos\big((j-k)x\big)-\cos\big((j+k)x\big)}{2}\mathrm{d}x = \frac{\pi\delta_{jk}}{2},$$

so $\{v_j\}$ are **orthogonal** with respect to the L^2-inner product. Taking analogy from the matrix-vector case, if $f=\sum_{j=1}^\infty \beta_j v_j$ is true, then

$$\langle v_k, f\rangle = \sum_{j=1}^\infty \beta_j\langle v_k, v_j\rangle = \frac{\pi\beta_k}{2} \qquad \Rightarrow \qquad \beta_k = \frac{2}{\pi}\langle f, v_k\rangle = \frac{2}{\pi}\int_0^\pi f v_k.$$

Note that this also gives the **uniqueness** of $\{\beta_j\}$. We commuted the inner product with the infinite sum without justification. The same procedure also would give the affirmative answer to Question 4.

Sine series

Now we give a meaning to the equality $f = \sum_{j=1}^{\infty} \beta_j v_j$. Define the L^2-**norm**

$$\|w\|^2 = \int_0^{\pi} |w(x)|^2 \mathrm{d}x.$$

We say that $f = \sum_{j=1}^{\infty} \beta_j v_j$ **in the L^2-sense** if

$$\left\| f - \sum_{j=1}^{n} \beta_j v_j \right\| \to 0 \qquad \text{as} \quad n \to \infty.$$

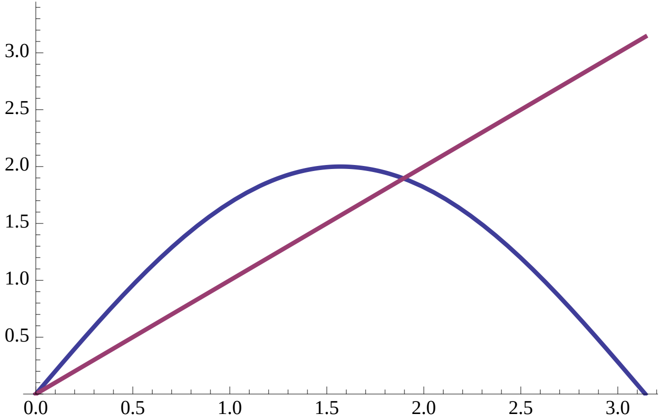
We take the followings facts as given.

- Any function f with $\|f\| < \infty$ satisfies $f = \sum_{j=1}^{\infty} \beta_j v_j$ in the L^2-sense, with the unique coefficients $\beta_j = \frac{2}{\pi} \langle f, v_j \rangle$.
- $\left(\sum_{j=1}^{\infty} \xi_j v_j \right)'' = \sum_{j=1}^{\infty} \xi_j \left(v_j \right)''$.

Example

$f(x) = x$

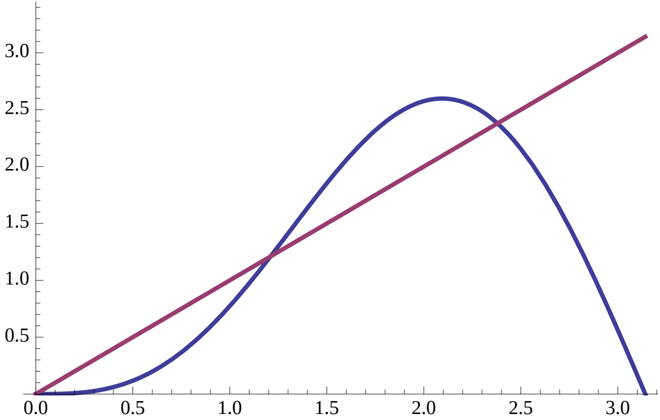
1 term:



Example

$f(x) = x$

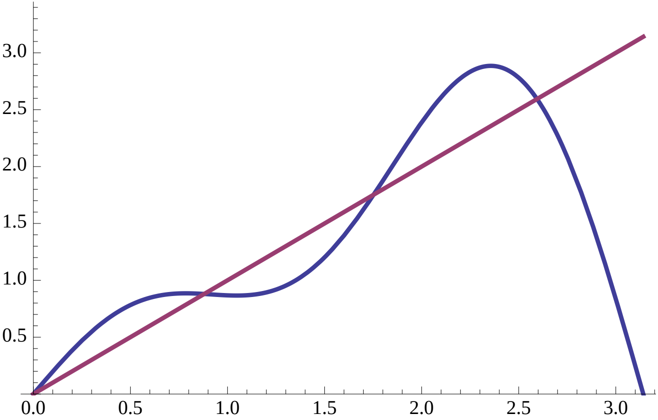
2 terms:



Example

$f(x) = x$

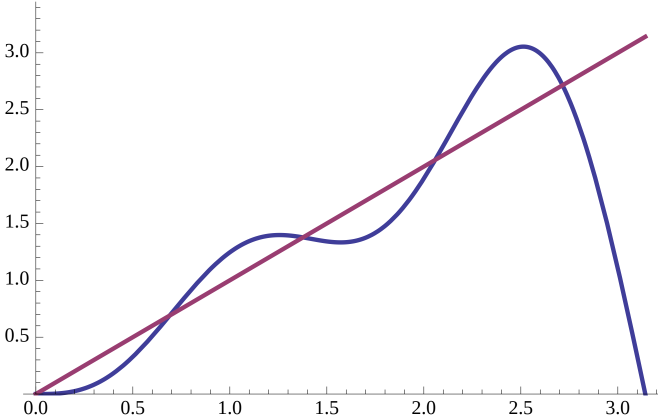
3 terms:



Example

$f(x) = x$

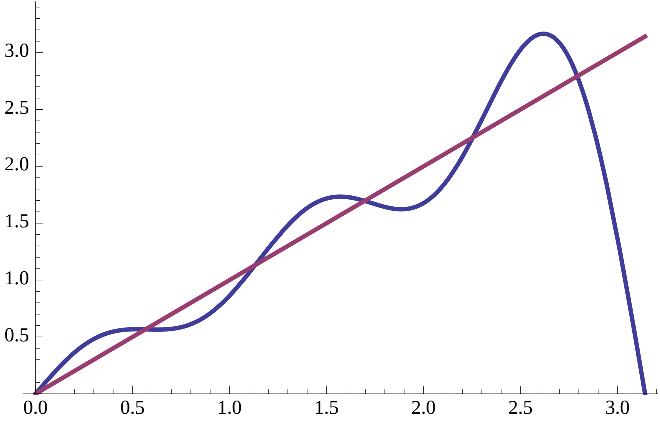
4 terms:



Example

$f(x) = x$

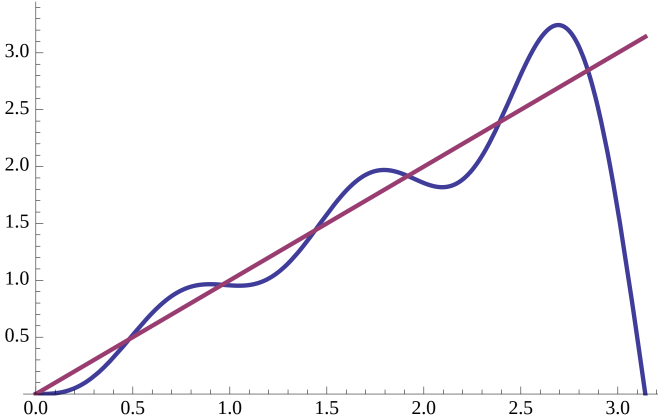
5 terms:



Example

$f(x) = x$

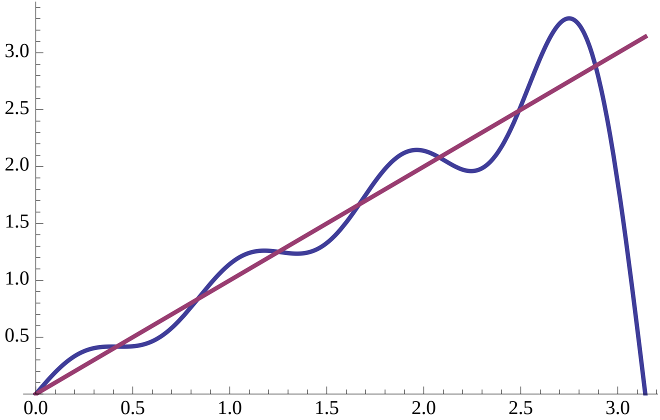
6 terms:



Example

$f(x) = x$

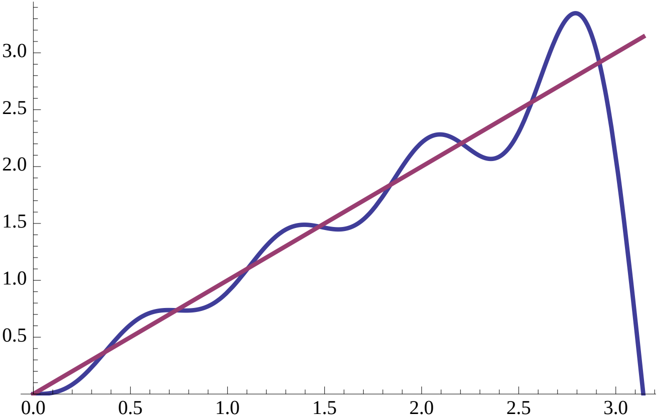
7 terms:



Example

$f(x) = x$

8 terms:



$f(x) = x$

1, 4, 8 terms: