Lecture 17: Implicit methods, classification of second order PDEs

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To allow large time steps, one needs to use **implicit methods**, such as

$$\frac{u_{i,k+1} - u_{i,k}}{\tau} = \frac{u_{i+1,k+1} + u_{i+1,k+1} - 2u_{i,k+1}}{h^2}, \qquad u_{i,0} = f(ih).$$

This can be rewritten as

$$\left(1 + \frac{2\tau}{h^2}\right) u_{i,k+1} - \frac{\tau}{h^2} u_{i-1,k+1} - \frac{\tau}{h^2} u_{i+1,k+1} = u_{i,k}, \qquad u_{i,0} = f(ih).$$

So, in order to find $u_{i,k+1}$ one needs to solve a system of linear equations.

This scheme is unconditionally stable. In general, implicit methods are more expensive than explicit methods due to matrix inversion. But in 1D, the overhead is negligible, because one can solve tridiagonal system in $O(n)$ time.

Discrete maximum principle

Recall the implicit method from the preceding page:

$$(1+2c)\,u_{i,k+1} - c u_{i-1,k+1} - c u_{i+1,k+1} = u_{i,k},$$

where $c = \tau / h^2 > 0$. This can be written as

$$u_{i,k+1} = \frac{c u_{i-1,k+1} + u_{i,k} + c u_{i+1,k+1}}{1+2c}.$$

Theorem

Fix some K, and let $M = \max_{k \le K} \max_i u_{i,k}$. If $u_{j,n} = M$ for some interior node (j,n) with $n \le K$, $u_{i,k}$ must be constant for all i and all $k \le K$.

Proof.

Suppose that $M = u_{i,k+1}$ for some particular (interior) node $(i,k+1)$ with $k+1 \le K$. Then it must be the case that

$$u_{i-1,k+1} = u_{i,k} = u_{i+1,k+1} = M,$$

otherwise they cannot have M as their average. This reasoning "propagates" to make every $u_{i,k} = M$ for $k \le K$. □

Consider the general second order equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + F(u_x, u_y, u) = 0.$$

The first three terms are called the **principal terms**, and the behavior of the equation is largely determined by the coefficients A, B, and C. The terms represented by F are called the **lower order terms**. Note

- $A = C = 1$, $B = 0$ is the *Laplace equation* with lower order terms.
- $A = 1$, $C = -1$, $B = 0$ is the *wave equation* with lower order terms.
- $A = C = 0$, $B = 1$ is also the *wave equation* with lower order terms.
- $A = 1$, $B = C = 0$ is something like the *heat equation*, if there is a lower order term with u_y.

We want to perform a change of variables, and try to transform the above equation into one of these simple forms. Without loss of generality, let us assume that A and B are nonzero.

Change of variables

We consider the general linear change of variables

$$\xi = \alpha x + \beta y, \qquad \eta = \gamma x + \delta y.$$

We have

$$u_x = \alpha u_\xi + \gamma u_\eta, \qquad u_{xx} = \alpha^2 u_{\xi\xi} + \gamma^2 u_{\eta\eta} + 2\alpha\gamma u_{\xi\eta},$$

$$u_y = \beta u_\xi + \delta u_\eta, \qquad u_{yy} = \beta^2 u_{\xi\xi} + \delta^2 u_{\eta\eta} + 2\beta\delta u_{\xi\eta},$$

$$u_{xy} = \alpha\beta u_{\xi\xi} + \gamma\delta u_{\eta\eta} + (\beta\gamma + \alpha\delta) u_{\xi\eta},$$

and so

$$\begin{aligned} Au_{xx} + Bu_{xy} + Cu_{yy} &= \left(A\alpha^2 + B\alpha\beta + C\beta^2\right) u_{\xi\xi} \\ &\quad + \left(2A\alpha\gamma + B(\beta\gamma + \alpha\delta) + 2C\beta\delta\right) u_{\xi\eta} \\ &\quad + \left(A\gamma^2 + B\gamma\delta + C\delta^2\right) u_{\eta\eta} \\ &=: A' u_{\xi\xi} + B' u_{\xi\eta} + C' u_{\eta\eta}. \end{aligned}$$

Hyperbolic case

If $D := B^2 - 4AC > 0$, then we can make $A' = C' = 0$, by choosing

$$\beta = \delta = 1, \qquad \alpha = \frac{-B + \sqrt{D}}{2A}, \qquad \gamma = \frac{-B - \sqrt{D}}{2A}.$$

We have

$$B' = 2A\alpha\gamma + B(\gamma + \alpha) + 2C = 2A\frac{-(B^2 - D)}{4A^2} + B\frac{-2B}{2A} + 2C = -\frac{B^2}{A} \neq 0.$$

So the equation becomes of the form

$$u_{\xi\eta} + G(u_\xi, u_\eta, u) = 0,$$

which is a wave equation with lower order terms. This case (i.e., $D > 0$) is called **hyperbolic**. Note that once we have the above form, by changing to, e.g., $\xi + \eta$ and $\xi - \eta$, we can transform the equation into the form

$$u_{\xi\xi} - u_{\eta\eta} + H(u_\xi, u_\eta, u) = 0.$$

Parabolic case

If $D = 0$, then choose

$$\beta = \delta = 1, \qquad \gamma = -\frac{B}{2A}.$$

We have

$$A' = A\alpha^2 + B\alpha + C,$$
$$B' = -B\alpha + B\alpha - \frac{B^2}{2A} + 2C = -\frac{B^2 - 4AC}{2A} = 0.$$

The equation becomes of the form

$$u_{\xi\xi} + G(u_\xi, u_\eta, u) = 0,$$

which resembles the heat equation. This case (i.e., $D = 0$) is called **parabolic**.

Elliptic case

If $D<0$, then choose $\alpha=1$ and $\beta=0$. We have

$$A' = A, \qquad B' = 2A\gamma + B\delta.$$

We let $\delta = -2A\gamma/B$ so as to make $B'=0$. The remaining coefficient is

$$C' = A\gamma^2 - 2A\gamma^2 + \frac{4A^2\gamma^2 C}{B^2} = \frac{A\gamma^2}{B^2}\left(4AC - B^2\right) = \frac{A\gamma^2(-D)}{B^2}.$$

If we choose $\gamma = B/\sqrt{-D}$, we have $C' = A = A'$. The equation becomes of the form

$$u_{\xi\xi} + u_{\eta\eta} + G(u_\xi, u_\eta, u) = 0,$$

which is the Laplace equation with lower order terms. This case (i.e., $D<0$) is called **elliptic**.

Discriminant

Depending on the sign of $D = B^2 - 4AC$, we transformed the equation into wave-like, heat-like, or Laplace-like forms. Can these forms be transformed from one to another? We have

$$A' = A\alpha^2 + B\alpha\beta + C\beta^2 = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^T \begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

$$B' = 2A\alpha\gamma + B(\beta\gamma + \alpha\delta) + 2C\beta\delta = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^T \begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix},$$

$$C' = A\gamma^2 + B\gamma\delta + C\delta^2 = \begin{pmatrix}\gamma\\ \delta\end{pmatrix}^T \begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}\gamma\\ \delta\end{pmatrix},$$

which can be rewritten as

$$\begin{pmatrix}A' & B'/2\\ B'/2 & C'\end{pmatrix} = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}^T \begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}.$$

Therefore we have

$$-\frac{D'}{4} = \det\begin{pmatrix}A' & B'/2\\ B'/2 & C'\end{pmatrix} = (\alpha\delta - \beta\gamma)^2\left(-\frac{D}{4}\right),$$

so one cannot alter the sign of D by change of variables.

Classification of second order equations

We classify

$$Au_{xx} + Bu_{xy} + Cu_{yy} + F(u_x, u_y, u) = 0,$$

according to the sign of $D = B^2 - 4AC$:

- **Hyperbolic** $D > 0$: The canonical forms are
  $u_{\xi\xi} - u_{\eta\eta} + G(u_\xi, u_\eta, u) = 0$ and $u_{\xi\eta} + H(u_\xi, u_\eta, u) = 0$.
- **Parabolic** $D = 0$: The canonical form is $u_{\xi\xi} + G(u_\xi, u_\eta, u) = 0$.
- **Elliptic** $D < 0$: The canonical form is $u_{\xi\xi} + u_{\eta\eta} + G(u_\xi, u_\eta, u) = 0$.

This classification can be extended to the cases where

- more than two independent variables present.
- F depends on x, y, in particular, to **inhomogneeous** equations.
- F is nonlinear (this is called the **semilinear** case).
- A, B, C depend on x, y. In this case, the equation type may depend on x, y, but usually the coefficients are so that the equation has a single well-defined type.
- A, B, C depend on u (**quasilinear** equation). In this case, the equation type may depend on the behavior of u itself.