Lecture 16: Heat equation (continued)

Gantumur Tsogtgerel

Assistant professor of Mathematics

Math 319: Introduction to PDEs

McGill University, Montréal

Tuesday February 8, 2011

Questions

- What are the harmonic functions in 1 dimension?
- The linears: $u(x) = kx + m \Leftrightarrow u_{xx} = 0$.
- Suppose that you solve a linear system with $n \times n$ matrix by using **Gaussian elimination**, and let us say each elementary operation (addition, multiplication etc.) on numbers requires one unit of work. As n grows to get very large, how does the work grow for general matrices? Is it $O(n^2)$? $O(n^3)$? $O(2^n)$?
- $O(n^3)$.
- The same question, if the matrix is known to be **tridiagonal**?
- $O(n)$.

Convection-diffusion equation

The following is called the **convecion-diffusion equation**:

$$u_t + cu_x = u_{xx}.$$

Let us change to the *characteristic coordinates*

$$\xi = x - ct, \qquad \tau = t.$$

We have

$$u_x = u_\xi, \quad u_{xx} = u_{\xi\xi}, \quad u_t = -cu_\xi + u_\tau, \qquad \text{and so} \qquad u_t + cu_x = u_\tau.$$

Therefore the convection-diffusion equation is just the heat equation in the characteristic coordinates

$$u_\tau = u_{\xi\xi}.$$

Similarly, the *convection-reaction-diffusion* equation

$$u_t + cu_x = u_{xx} + \lambda u,$$

becomes

$$u_\tau = u_{\xi\xi} + \lambda u.$$

Let Ω be a domain, and consider the **initial-boundary value problem**

$$u_t = \Delta u, \qquad u(x,0) = f(x),\ x \in \Omega, \qquad u(x,t) = g(x),\ x \in \partial\Omega.$$

We expect that as $t \to \infty$, the function $u(x,t)$ settles down to some steady profile: $u(x,t) \to v(x)$ for some v. If this is the case, $u_t \to 0$, so v satisfies

$$\Delta v = 0, \qquad v(x) = g(x),\ x \in \partial\Omega.$$

If we define $w = u - v$, it satisfies

$$w_t = \Delta w, \qquad w(x,0) = f(x) - v(x),\ x \in \Omega, \qquad w(x,t) = 0,\ x \in \partial\Omega.$$

From physical intuition we can say that $w \to 0$ as $t \to \infty$. The steady states of the heat equation are harmonic functions, and we see that the Laplace equation could describe the steady-state temperature distribution in a body with given temperature at the boundary.

Forward time differencing: Explicit methods

Apply a finite difference scheme to

$$u_t = u_{xx}, \qquad u(x,0) = f(x),$$

with *time-step* τ and spacial *mesh-size* h:

$$\frac{u_{i,k+1} - u_{i,k}}{\tau} = \frac{u_{i-1,k} - 2u_{i,k} + u_{i+1,k}}{h^2}, \qquad u_{i,0} = f(ih),$$

or

$$u_{i,k+1} = \left(1 - \frac{2\tau}{h^2}\right) u_{i,k} + \frac{\tau}{h^2} u_{i-1,k} + \frac{\tau}{h^2} u_{i+1,k}, \qquad u_{i,0} = f(ih).$$

So, $u_{i,k+1}$ can be calculated directly from the data on the time level k. In particular, no matrix inversion is needed. Such formulas are called **explicit methods**.

This scheme is unstable unless the time step is so small that $\tau \leq h^2/2$.