Lecture 15: Heat equation on the real line

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Math 319: Introduction to PDEs

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Questions

- What is the derivative of

$$f(x) = \begin{Bmatrix} 0, & \text{for } x < 0 \\ x, & \text{for } x > 0 \end{Bmatrix} \quad ?$$

- $f' = \theta$, where θ is the *Heaviside step function*

$$\theta(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases}$$

- What is the derivative of θ?
- The delta "function" $\delta(x)$: $\theta' = \delta$.
- What is the value of $\displaystyle\int_0^\infty e^{-s^2} \mathrm{d}s$?
- $\displaystyle\int_0^\infty e^{-s^2} \mathrm{d}s = \frac{\sqrt{\pi}}{2}$.

Heat conduction

Let us consider the heat conduction problem in a homogeneous bar, that can be thought of as a 1D object. Imagine that the bar is divided into small segments of length h, and let x be the midpoint of one of those segments. Then the heat energy in that segment at the time moment t is

$$Q(t) = \sigma \rho s h u(x,t),$$

where σ is the specific heat, ρ is the density, s is the cross sectional area of the bar, and we assumed that u is constant thoughout the segment.

The heat transferred to the segment from its two neighbors in time interval τ is

$$F = ks\tau \frac{u(x-h,t)-u(x,t)}{h} + ks\tau \frac{u(x+h,t)-u(x,t)}{h},$$

where k is the heat conductivity. A combination of these gives

$$\begin{aligned} Q(t+\tau)-Q(t) &= \sigma\rho s h\big(u(x,t+\tau)-u(x,t)\big) \\ &= ks\tau\left(\frac{u(x+h,t)-u(x,t)}{h} - \frac{u(x-h,t)-u(x,t)}{h}\right). \end{aligned}$$

Heat equation

Rewriting

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{k}{\sigma\rho}\,\frac{\frac{u(x+h,t)-u(x,t)}{h} - \frac{u(x-h,t)-u(x,t)}{h}}{h},$$

and sending $h,\tau \to 0$, we get the **heat equation**

$$u_t = \kappa u_{xx},$$

also known as the *diffusion equation*.

The heat equation is used to model the diffusion of heat, chemicals, and other quantities.

Note that if $u(x,t)$ satisfies $u_t = \kappa u_{xx}$, then $v(x,t) = u(x,t/\kappa)$ satisfies

$$v_t = v_{xx}.$$

Self similar solutions

Note also that if $u(x,t)$ is a solution, so is $v(x,t) = u(\lambda x, \lambda^2 t)$ for any λ. Therefore it is natural to look for solutions of the form $u(x,t) = w(\frac{x^2}{t})$. We have

$$u_t(x,t) = w'\left(\frac{x^2}{t}\right)\cdot\left(-\frac{x^2}{t^2}\right), \qquad u_x(x,t) = w'\left(\frac{x^2}{t}\right)\cdot\left(\frac{2x}{t}\right),$$

$$u_{xx}(x,t) = w''\left(\frac{x^2}{t}\right)\cdot\left(\frac{4x^2}{t^2}\right) + w'\left(\frac{x^2}{t}\right)\cdot\left(\frac{2}{t}\right).$$

Upon imposing $u_t = u_{xx}$, this leads to

$$4w''(\xi) + (\frac{2}{\xi} + 1)w'(\xi) = 0,$$

where $\xi = \frac{x^2}{t}$. Solving for w' gives

$$w'(\xi) = C\xi^{-1/2}e^{-\xi/4},$$

and finally

$$w(\xi) = C\int_0^\xi \frac{e^{-r/4}}{\sqrt{r}}\mathrm{d}r + C_1.$$

Self similar solutions

By the substitutions $r = y^2$ and $y = 2s$, we get

$$w(\xi) = C\int_0^{\sqrt{\xi}} e^{-y^2/4}\mathrm{d}y + C_1 = C\int_0^{\sqrt{\xi}/2} e^{-s^2}\mathrm{d}s + C_1.$$

We derived

$$u(x,t) = C\int_0^{x/\sqrt{4t}} e^{-s^2}\mathrm{d}s + C_1,$$

as a solution of the heat equation. Consider the initial condition

$$u(x,0) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases}$$

For $x > 0$ and $x < 0$ respectively, letting $t \to 0$, we infer

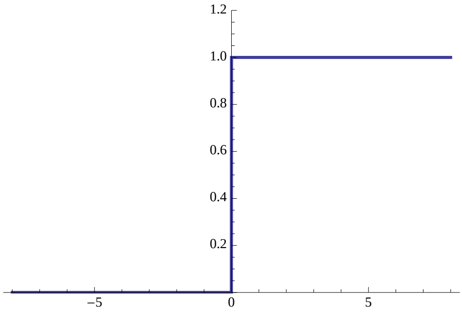
$$1 = C\int_0^{\infty} e^{-s^2}\mathrm{d}s + C_1 = \sqrt{\pi}C/2 + C_1, \qquad 0 = C\int_0^{-\infty} e^{-s^2}\mathrm{d}s + C_1 = -\sqrt{\pi}C/2 + C_1.$$

Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 0$:



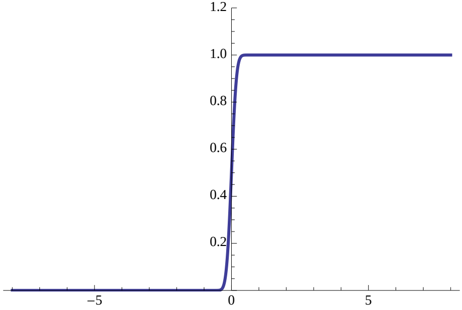
An idealized heat transfer problem

Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2} \mathrm{erf}\left(\frac{x}{\sqrt{4t}} \right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 0.01$:

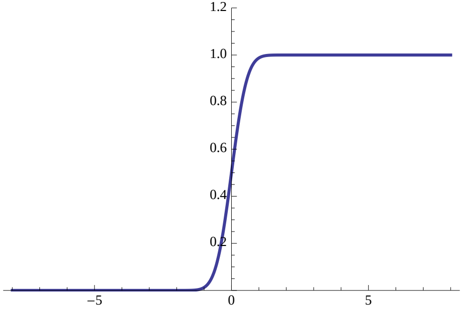


Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2}\,\mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 0.1$:

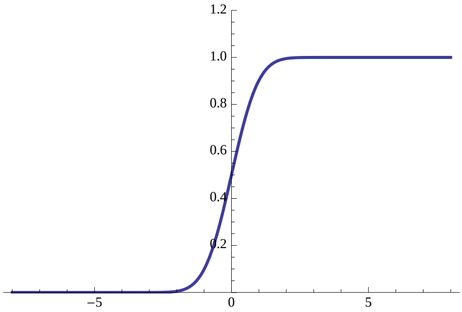


Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 0.3$:



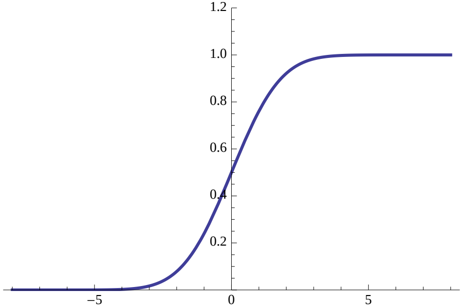
An idealized heat transfer problem

Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2}\,\mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 1$:

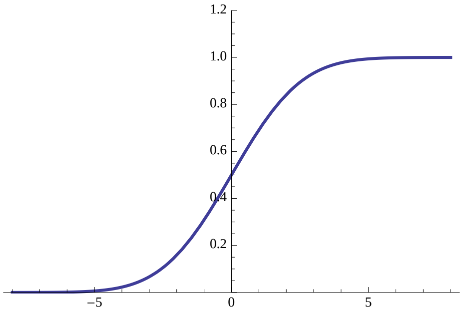


Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2} \mathrm{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 2$:



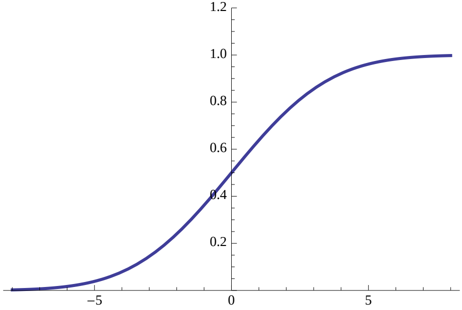
An idealized heat transfer problem

Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 4$:

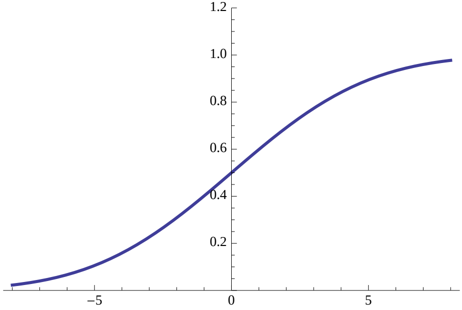


Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2} \mathrm{erf}\left(\frac{x}{\sqrt{4t}} \right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 8$:



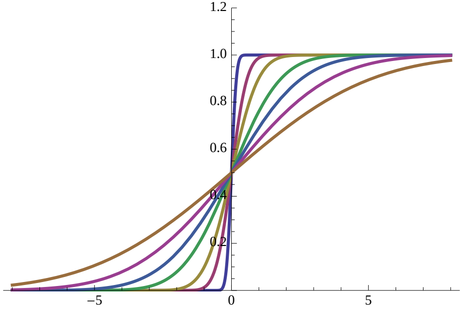
An idealized heat transfer problem

Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2}\,\mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

Time $t = 0, 0.01, 0.1, 0.3, 1, 2, 4, 8$:

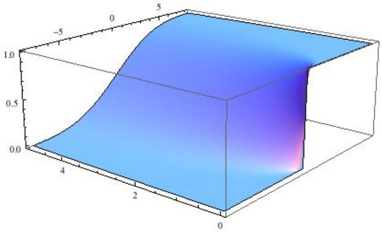


Finally, we have

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2}\,\mathrm{d}s \equiv \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4t}}\right),$$

solving the heat equation with the initial condition given by the Heaviside step function.

3D plot:



The fundamental solution

If we differentiate

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-s^2} \mathrm{d}s,$$

with respect to x, we find

$$G(x,t) := u_x(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)},$$

which also solves the heat equation. This solution is called the **fundamental solution** of the heat equation.

Note that $G(0,t) \to \infty$ as $t \to 0$, and $G(x,t) \to 0$ as $t \to 0$ if $x \neq 0$. Note also that for any $t > 0$

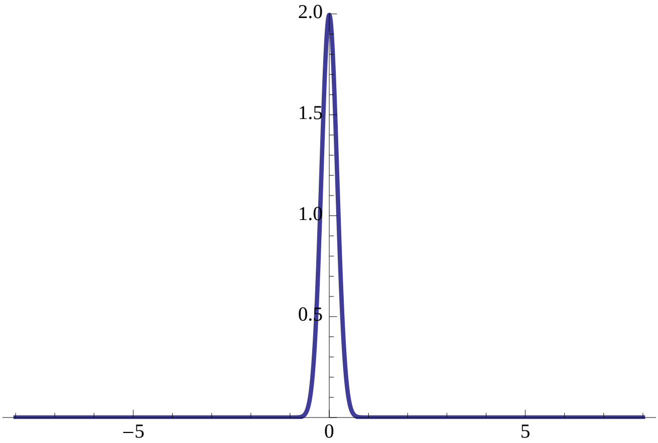
$$\int_{-\infty}^{\infty} G(x,t) \mathrm{d}x = 1,$$

so the initial condition for the fundamental solution is the delta function concentrated at 0.

The fundamental solution

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

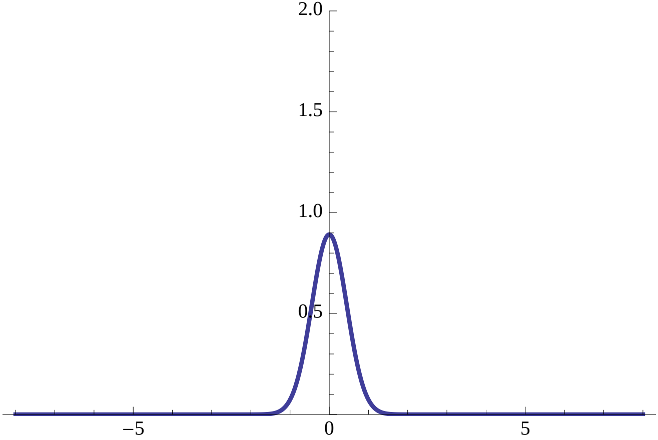
Time $t = 0.02$:



The fundamental solution

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

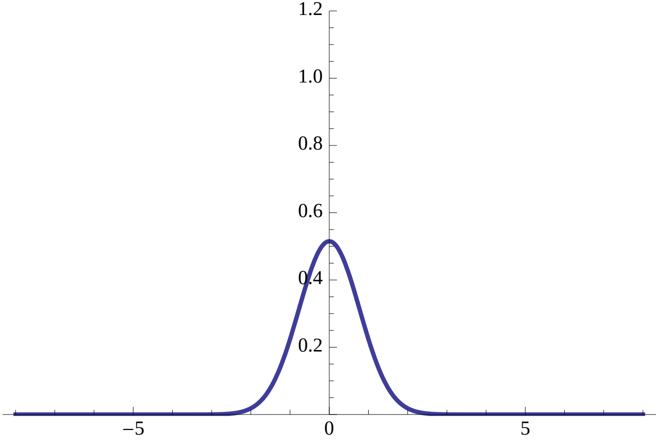
Time $t = 0.1$:



The fundamental solution

$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$

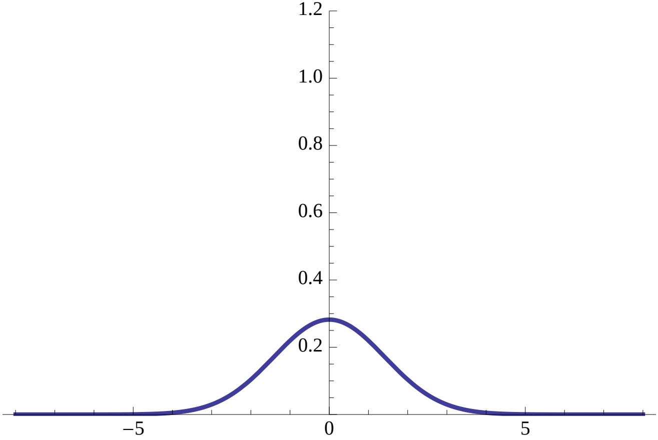
Time $t = 0.3$:



The fundamental solution

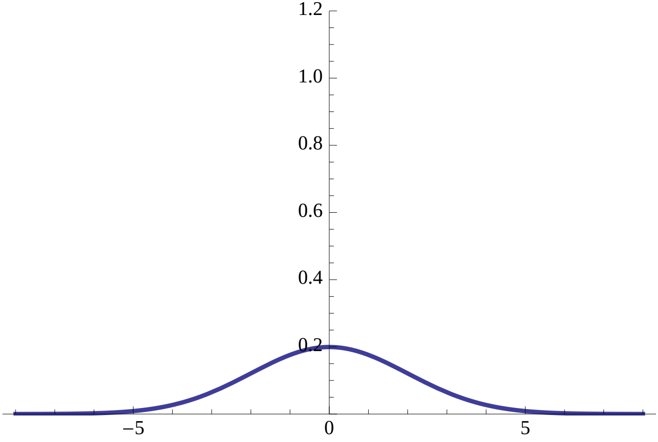
$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

Time $t = 1$:



$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$

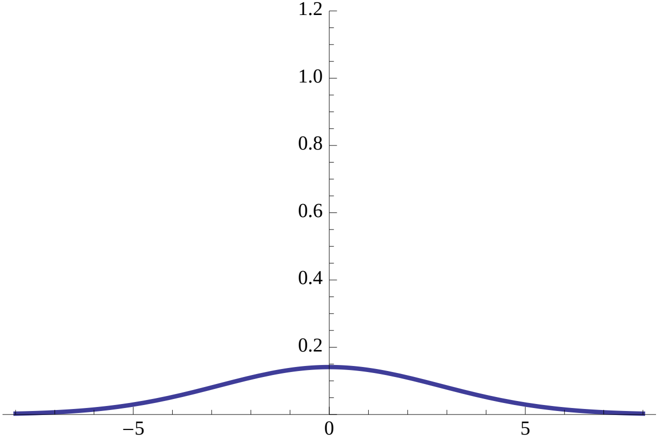
Time $t = 2$:



The fundamental solution

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

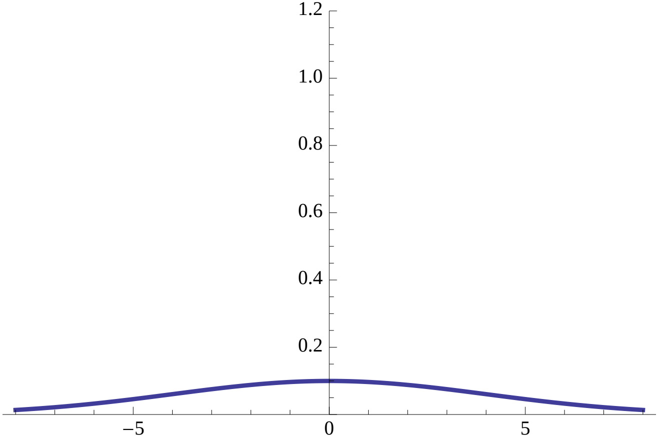
Time $t = 4$:



The fundamental solution

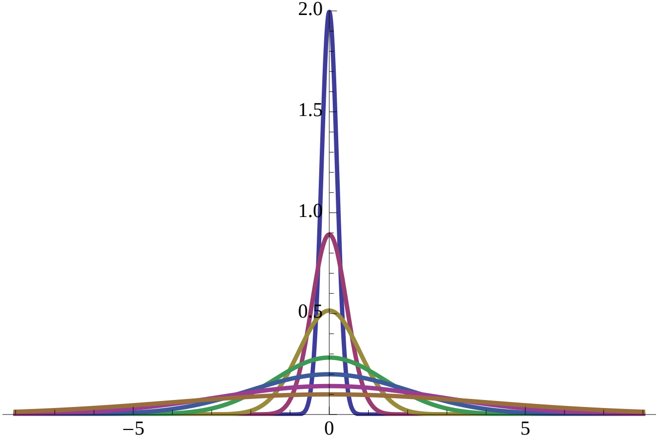
$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

Time $t = 8$:



$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

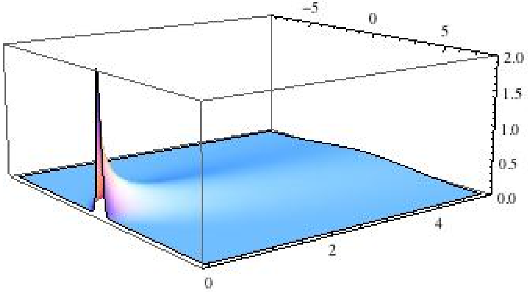
Time $t = 0.02, 0.1, 0.3, 1, 2, 4, 8$:



The fundamental solution

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

3D plot:



For general initial condition $u(x,0) = f(x)$, the heat equation is solved by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y) \mathrm{d}y = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-|x-y|^2/(4t)} f(y) \mathrm{d}y.$$

Observe that

- $u(x,t) \to 0$ like $\frac{1}{\sqrt{t}}$ as $t \to \infty$.
- information propagates with infinite speed.

Although we do not prove here, it is true that

- $u(x,t)$ is infinitely smooth as a function of x, for $t > 0$.
- $u(x,t)$ behaves badly if $t < 0$, even for reasonable choices of f.