Lecture 14: Spherical waves and wave energy

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Let us look for radial solutions of the 3D wave equation, i.e., consider

$$u_{tt} - \Delta u = 0, \qquad u(r,0) = f(r), \qquad u_t(r,0) = g(r),$$

where r is the radial coordinate. We have

$$\begin{aligned}\Delta u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}\\ &= \frac{2}{r}u_r + u_{rr} = \frac{1}{r}(u_r + u_r + ru_{rr}) = \frac{1}{r}(u + ru_r)_r = \frac{1}{r}(ru)_{rr},\end{aligned}$$

so the wave equation becomes

$$\frac{1}{r}(ru)_{rr} - u_{tt} = 0 \qquad \Rightarrow \qquad (ru)_{rr} - (ru)_{tt} = 0,$$

hence $v(r,t) = ru(r,t)$ satisfies the 1D wave equation.

D'Alamber's formula gives

$$ru(r,t) = \frac{v(r+t,0)+v(r-t,0)}{2} + \frac{1}{2}\int_{r-t}^{r+t} v_t(s,0)\mathrm{d}s$$
$$= \frac{(r+t)f(r+t)+(r-t)f(r-t)}{2} + \frac{1}{2}\int_{r-t}^{r+t} sg(s)\mathrm{d}s,$$

and so

$$u(r,t) = \frac{f(r+t)+f(r-t)}{2} + t\cdot\frac{f(r+t)-f(r-t)}{2r} + \frac{1}{2r}\int_{r-t}^{r+t} sg(s)\mathrm{d}s.$$

We are interested in the solution at $r=0$, in particular, whether or not $u(0,t)$ is finite. Since f is even, for the first term we have

$$f(t)+f(-t) = f(t)+f(t) = 2f(t).$$

We have, for the second term

$$\frac{f(r+t)-f(r-t)}{2r}=\frac{f(r+t)-f(t-r)}{2r}\to f'(t)\qquad \text{as}\quad r\to 0,$$

and for the third term

$$\int_{r-t}^{r+t} sg(s)\,\mathrm{d}s=\int_{0}^{r+t}+\int_{r-t}^{0}=\int_{0}^{r+t}-\int_{0}^{t-r}=\int_{t-r}^{t+r} sg(s)\,\mathrm{d}s=2r\cdot tg(t)+O(r^2).$$

Finally

$$u(0,t)=f(t)+tf'(t)+tg(t)=\big(tf(t)\big)_t+tg(t).$$

Alternative derivation of Poisson's formula

For any function u, define its spherical average

$$\overline{u}(r,t) = \frac{1}{4\pi r^2} \int\limits_{|x|=r} u(x,t)\mathrm{d}S_x.$$

If u satisfies the wave equation, so does $\overline{u}$. We have

$$u(0,t) = \overline{u}(0,t) = \big(t\overline{u}(t,0)\big)_t + t\overline{u}_t(t,0).$$

Then noting that

$$\overline{u}(t,0) = \frac{1}{4\pi t^2} \int\limits_{|x|=t} f(x)\mathrm{d}S_x,$$

and

$$\overline{u}_t(t,0) = \frac{1}{4\pi t^2} \int\limits_{|x|=t} g(x)\mathrm{d}S_x,$$

Poisson's formula follows.

Energy

Let $u(x,y,t)$ be the solution of the 2D wave equation, and consider a (right) cone with vertex at some (x_0,y_0,t_0), $t_0>0$, and with circular base of radius t_0 in the initial plane surface $t=0$. Let $0\leq T\leq t_0$, and denote the intersection of the solid cone with the plane $t=T$ by D_T. In particular, D_0 is the base or the "bottom surface" of the cone.
We want to prove

$$\mathscr{E}(T):=\int_{D_T} u_x^2+u_y^2+u_t^2 \leq \int_{D_0} u_x^2+u_y^2+u_t^2 \equiv \mathscr{E}(0),$$

for the wave equation in 2D. We will use the identity

$$2u_t(u_{xx}+u_{yy}-u_{tt}) = (2u_tu_x)_x+(2u_tu_y)_y-(u_x^2+u_y^2+u_t^2)_t =: \operatorname{div} W.$$

Since $u_{xx}+u_{yy}-u_{tt}=0$, with Ω being the part of the solid cone that lies between D_0 and D_T, and C being the lateral surface of Ω, we have

$$0=\int_\Omega 2u_t(u_{xx}+u_{yy}-u_{tt}) = \int_\Omega \operatorname{div} W = \int_{\partial\Omega} W = -\int_{D_T}(u_x^2+u_y^2+u_t^2)$$
$$+\int_{D_0}(u_x^2+u_y^2+u_t^2)+\int_C 2u_tu_xn_x+2u_tu_yn_y-(u_x^2+u_y^2+u_t^2)n_t,$$

where (n_x,n_y,n_t) is the unit normal vector of the conical surface C.

From $n_x^2+n_y^2=n_t^2$ and $n_x^2+n_y^2+n_t^2=1$, we have $n_t^2=n_x^2+n_y^2=\frac{1}{2}$.
Let us rearrange the integrand of the third integral as

$$\begin{aligned}2u_tu_xn_x+2u_tu_yn_y-(u_x^2+u_y^2+u_t^2)n_t \\ =2u_tu_xn_x+2u_tu_yn_y-(u_x^2+u_y^2)n_t-u_t^2(n_x^2+n_y^2)\sqrt{2}.\end{aligned}$$

Now we complete the squares

$$\begin{aligned}2u_tu_xn_x-u_x^2n_t-\sqrt{2}u_t^2n_x^2&=2\sqrt{2}u_tu_xn_xn_t-\sqrt{2}u_x^2n_t^2-\sqrt{2}u_t^2n_x^2\\&=-\sqrt{2}(u_xn_t-u_tn_x)^2.\end{aligned}$$

Using this, we get

$$\begin{aligned}\int_{D_T}(u_x^2+u_y^2+u_t^2)&=\int_{D_0}(u_x^2+u_y^2+u_t^2)-\sqrt{2}\int_C(u_xn_t-u_tn_x)^2+(u_yn_t-u_tn_y)^2\\&\leq\int_{D_0}(u_x^2+u_y^2+u_t^2).\end{aligned}$$

We have

$$\int_{D_T} (u_x^2 + u_y^2 + u_t^2) \leq \int_{D_0} (u_x^2 + u_y^2 + u_t^2).$$

So if $u = 0$ and $u_t = 0$ on D_0, then $u_x = u_y = u_t = 0$ at any point in Ω. Hence u is constant in Ω. Since $u = 0$ at D_0, $u = 0$ in Ω. By considering the difference between two supposed solutions of the Cauchy problem, we conclude

> *The solution u of the wave equation in the solid cone Ω is uniquely determined by u and u_t at the initial surface D_0.*
>
> *In particular, the Cauchy problem for the wave equation has a unique solution.*