Lecture 13: Waves in space and on the plane

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Maxwell's equations in vacuum are

$$\frac{\partial E}{\partial t} = c\nabla \times B, \qquad \nabla \cdot E = 0, \qquad \frac{\partial B}{\partial t} = -c\nabla \times E, \qquad \nabla \cdot B = 0,$$

where E is the electric field, and B is the magnetic field. We have

$$\frac{\partial^2 E}{\partial t^2} = c\nabla \times \frac{\partial B}{\partial t} = -c^2\nabla \times \nabla \times E = c^2\big(\Delta E - \nabla(\nabla \cdot E)\big) = c^2\Delta E.$$

So the electric field satisfies the wave equation

$$E_{tt} - c^2\Delta E = 0.$$

The same holds for the magnetic field.

Wave equation

The Cauchy problem for the wave equation in 3D is

$$u_{tt} - \Delta u = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

where $u(x,t)$ is the unknown solution $\big(x \in \mathbb{R}^3,\ t > 0\big)$.
Let $\phi(x)$ be a function defined on $\mathbb{R}^3$. Then we define

$$u(x,t) = \frac{1}{4\pi t} \int_{|y-x|=t} \phi(y) \mathrm{d}S_y = \frac{t}{4\pi} \int_{|y|=1} \phi(x+yt) \mathrm{d}S_y,$$

and claim that u satisfies the wave equation. Note that $u(x,0) = 0$.
We have

$$\Delta u = \frac{t}{4\pi} \int_{|y|=1} \Delta\phi(x+yt) \mathrm{d}S_y = \frac{1}{4\pi t} \int_{|y-x|=t} \Delta\phi(y) \mathrm{d}S_y.$$

$$
\begin{aligned}
u_t(x,t) &= \frac{1}{4\pi}\int_{|y|=1}\phi(x+yt)\,\mathrm{d}S_y + \frac{t}{4\pi}\int_{|y|=1}\nabla\phi(x+yt)\cdot y\,\mathrm{d}S_y \\
&= \frac{u(x,t)}{t} + \frac{1}{4\pi t}\int_{|y-x|=t}\nabla\phi(y)\cdot n\,\mathrm{d}S_y \\
&= \frac{u(x,t)}{t} + \frac{1}{4\pi t}\int_{|y-x|\le t}\Delta\phi(y)\,\mathrm{d}S_y =: \frac{u(x,t)}{t} + \frac{1}{4\pi t}\Phi(x,t).
\end{aligned}
$$

Note that $u_t(x,0)=\phi(x)$. For the second derivative

$$
u_{tt}(x,t) = \frac{tu_t-u}{t^2} + \frac{t\Phi_t-\Phi}{4\pi t^2} = \frac{1}{t}\left(u_t - \frac{u}{t} - \frac{1}{4\pi t}\Phi\right) + \frac{1}{4\pi t}\Phi_t = \frac{1}{4\pi t}\Phi_t.
$$

Note that $u_{tt}(x,0)=0$.

$$
\Phi_t = \frac{\partial}{\partial t}\int_{|y-x|\le t}\Delta\phi(y)\,\mathrm{d}S_y = \int_{|y-x|=t}\Delta\phi(y)\,\mathrm{d}S_y.
$$

So $\Delta u = u_{tt}$, $u(x,0)=0$, and $u_t(x,0)=\phi(x)$.

Poisson's formula

Since $u_{tt} - \Delta u = 0$, we have

$$0 = \frac{\partial}{\partial t}(u_{tt} - \Delta u) = (u_t)_{tt} - \Delta(u_t),$$

so $v = u_t$ satisfies $v_{tt} - \Delta v = 0$, $v(x,0) = \phi(x)$, and $v_t(x,0) = 0$. Hence

$$u(x,t) = \frac{1}{4\pi t}\int_{|y-x|=t} g(y)\mathrm{d}S_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=t} f(y)\mathrm{d}S_y\right),$$

satisfies $u_{tt} - \Delta u = 0$, $u(x,0) = f(x)$, and $u_t(x,0) = g(x)$. This formula is due to Poisson (1819), and often called also Kirchhoff's formula.
Introducing the spherical average

$$M_t[\phi](x) = \frac{1}{4\pi t^2}\int_{|y-x|=t} \phi(y)\mathrm{d}S_y,$$

Poisson's formula can be written as

$$u = tM_t[g] + \frac{\partial}{\partial t}(tM_t[f]).$$

Method of descent: Waves in 2D

Recall

$$u(x,t) = \frac{1}{4\pi t}\int_{|y-x|=t} g(y)\,\mathrm{d}S_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=t} f(y)\,\mathrm{d}S_y\right),$$

satisfies $u_{tt} - \Delta u = 0$, $u(x,0) = f(x)$, and $u_t(x,0) = g(x)$. If $f(x) = f(x_1, x_2, x_2)$ and $g(x) = g(x_1, x_2, x_3)$ are independent of x_3, then u defined above will also be independent of x_3, and moreover satisfy the 2-dimensional initial value problem

$$u_{tt} - \Delta_2 u = 0, \qquad u(\xi, 0) = f(\xi), \qquad u_t(\xi, 0) = g(\xi),$$

where Δ_2 is the 2-dimensional Laplacian, and $\xi \in \mathbb{R}^2$.
Explicit calculation gives

$$u(\xi,t) = \frac{1}{4\pi t}\int_{|\eta-\xi|\le t} \frac{g(\eta)\,\mathrm{d}\eta}{\sqrt{t^2 - |\eta-\xi|^2}} + \frac{\partial}{\partial t}\left(\int_{|\eta-\xi|\le t} \frac{f(\eta)\,\mathrm{d}\eta}{\sqrt{t^2 - |\eta-\xi|^2}}\right).$$