Lecture 12: Method of characteristics and the Courant-Friedrichs-Lewy condition

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Consider the initial value problem for the advection equation

$$u_t + c(x,t)\,u_x = 0, \qquad u(x,0) = f(x),$$

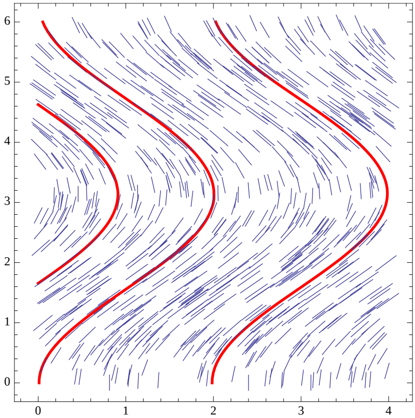
where the velocity $c(x,t)$ can vary in space and time.

Example: $c(x,t) = \sin t$

Characteristic curves satisfy

$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = c(x,t),$$

and $u(x(t),t)$ is constant along them.



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Example: $c(x,t) = x$

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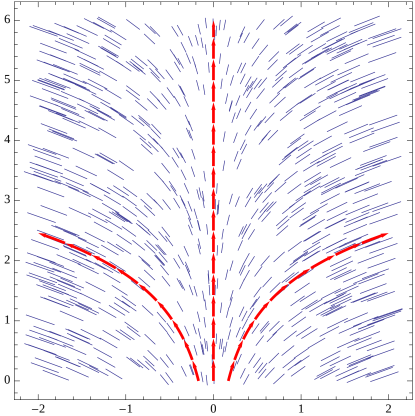

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$$u_t + c(x,t)\,u_x = 0, \qquad u(x,0) = f(x),$$

where the velocity $c(x,t)$ can vary in space and time.

Example: $c(x,t) = \cos x + \sin t$

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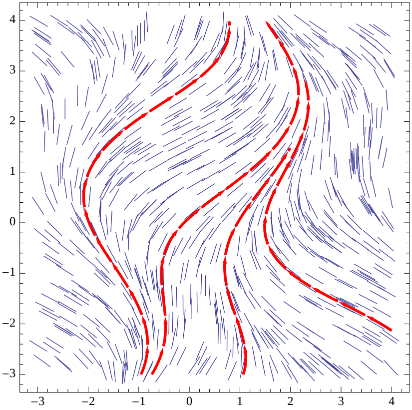

Method of characteristics

Let $c(x,t) = \sin t$. The characteristics are given by

$$\dot{x} = c(x,t) \qquad \Leftrightarrow \qquad x(t) = -\cos t + C = x_0 + 1 - \cos t,$$

where $x_0 = x(0)$.

$$x_0 = x + \cos t - 1, \qquad \Rightarrow \qquad u(x,t) = f(x + \cos t - 1).$$

Now consider $c(x,t) = x$. We have

$$x(t) = x_0 e^t, \qquad x_0 = x(t)e^{-t}, \qquad u(x,t) = f(xe^{-t}).$$

In general, suppose that the characteristics are given by $x(t) = h(x_0, t)$. Then if we can find $x_0 = q(x(t), t)$, we would have

$$u(x,t) = f(q(x,t)).$$

Linear hyperbolic systems

Consider the system

$$u_t + Au_x = 0,$$

where A is an $n \times n$ matrix, possibly depending on x and t. Suppose that $A = P^{-1}DP$ with a diagonal matrix D. Note that P and D can depend on x and t. We have

$$u_t + P^{-1}DPu_x = 0,$$

and after multiplying by P from the left

$$Pu_t + DPu_x = 0.$$

Upon introducing $q = Pu$, the new variable q satisfies

$$q_t + Dq_x = 0.$$

In general, we have n characteristic curves, corresponding to the eigenvalues $\lambda_i(x,t)$ of $A(x,t)$.

Domain of dependence, range of influence

Given a point (x,t), trace all the characteristic curves emanating from (x,t) *backward* in time. The area “under” these curves is the **domain of dependence** of (x,t). This is the set of points that can influence the point (x,t).

Now trace all the characteristic curves emanating from (x,t) *forward* in time. The area “above” these curves is the **range of influence** of (x,t). This is the set of points that can be influenced by the point (x,t).

Finite differences

Let us discretize

$$u_t + cu_x = 0, \qquad u(x,0) = f(x),$$

by a finite difference scheme, with *time-step* τ and spacial *mesh-size* h:

$$\frac{u_{i,k+1} - u_{i,k}}{\tau} + c\frac{u_{i+1,k} - u_{i,k}}{h} = 0, \qquad u_{i,0} = f(ih),$$

or

$$u_{i,k+1} = \left(1 + \frac{c\tau}{h}\right) u_{i,k} - \frac{c\tau}{h} u_{i+1,k}, \qquad u_{i,0} = f(ih).$$

So, $u_{i,k+1}$ can be calculated directly from the data on the **time level** k. In particular, no matrix inversion is needed.

If $c < 0$ then the above scheme is called an **upwind scheme**. If $c > 0$, an upwind scheme would use $u_{i-1,k}$ instead of $u_{i+1,k}$. "Downwind" scheme is never used.

Suppose that

$$u_{i,k+1} = F(u_{i-n,k}, \dots, u_{i+m,k}),$$

is a discretization of a hyperbolic system, on a grid with time-step τ and spacial mesh-size h. We say that the scheme **converges**, if for all i and k, $u_{i,k} \to u(ih, k\tau)$ as we send both h and τ to 0 in some fashion.

> *A necessary condition for the scheme to converge is that the line segment connecting the points $(ih - nh, k\tau)$ and $(ih + mh, k\tau)$ contains the domain of dependence of the point $(ih, k\tau + \tau)$ intersected with the line $t = k\tau$.*

This is called the Courant-Friedrichs-Lewy (CFL) condition. For our earlier example, the CFL condition is

$$h \geq c\tau, \qquad \text{or} \qquad \frac{c\tau}{h} \leq 1.$$