Lecture 11: D'Alambert's solution

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Math 319: Introduction to PDEs

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Where are we?

Laplace: $u_{xx} + u_{yy} = 0$, Poisson: $u_{xx} + u_{yy} = f$

- Stationary phenomena
- Boundary value (Dirichlet) problem: u is given at the boundary
- There exists a solution, and it is unique
- Explicitly solvable if the domain is the entire space
- Method of electrostatic images: works for very simple domains, and constant boundary conditions
- Finite differences: a numerical method for approximate solution

Wave: $u_{tt} - u_{xx} = 0$, advection: $u_t + u_x = 0$

- Transport phenomena
- Initial value (Cauchy) problem: some information is given at the initial time moment
- Characteristic coordinates are best suited
- D'Alambert's solution for the wave equation

Heat: $u_t - u_{xx} = 0$ (later)

D'Alambert's formula

Recall that the solution of the initial value problem for the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

is given by D'Alambert's formula

$$u(x,t) = \tfrac{1}{2} f(x+ct) + \tfrac{1}{2} f(x-ct) + \tfrac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

We can calculate the time derivative of u as

$$u_t(x,t) = \tfrac{c}{2} f'(x+ct) - \tfrac{c}{2} f'(x-ct) + \tfrac{1}{2} g(x+ct) + \tfrac{1}{2} g(x-ct).$$

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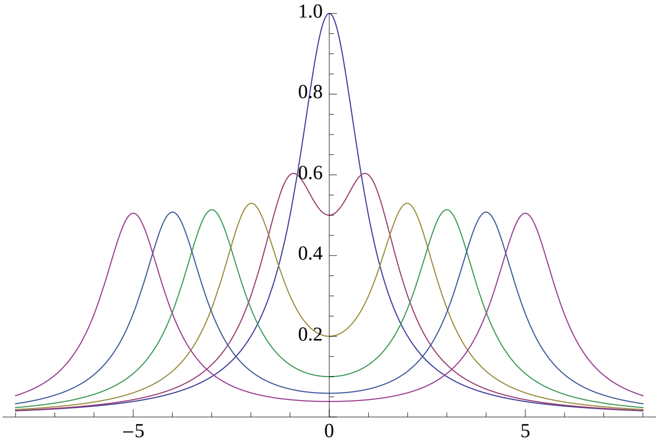
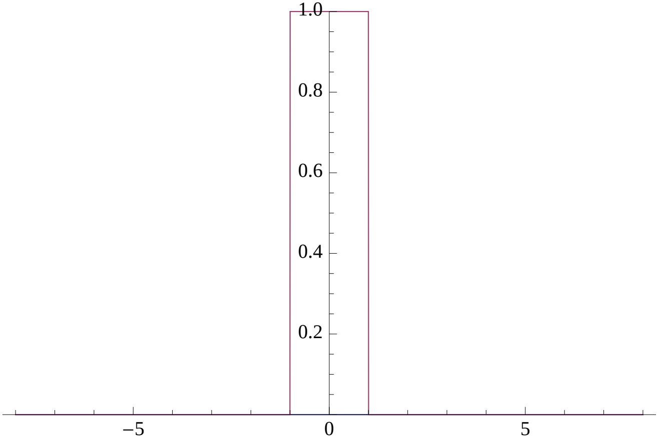
Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 0, \ldots, 5$:



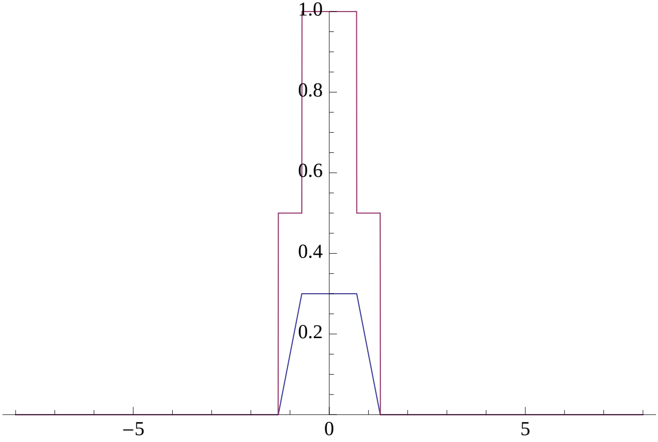
Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. $t = 0$:



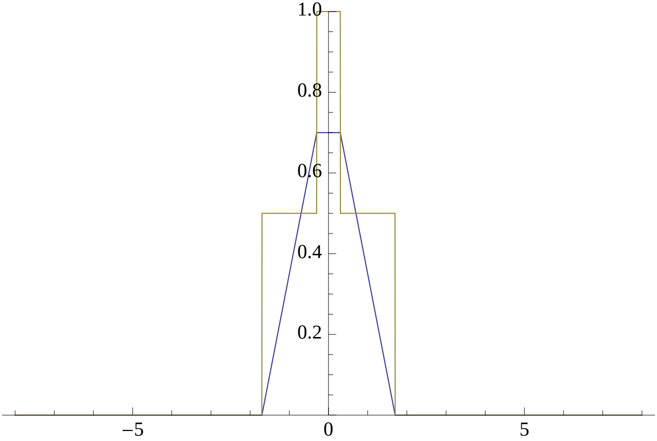
Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 0.3$:



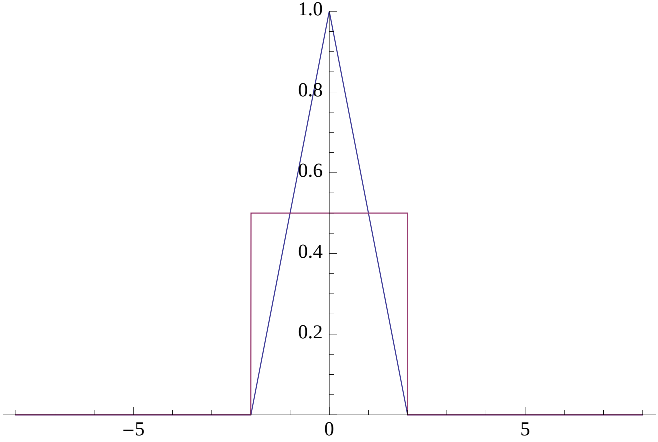
Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 0.7$:



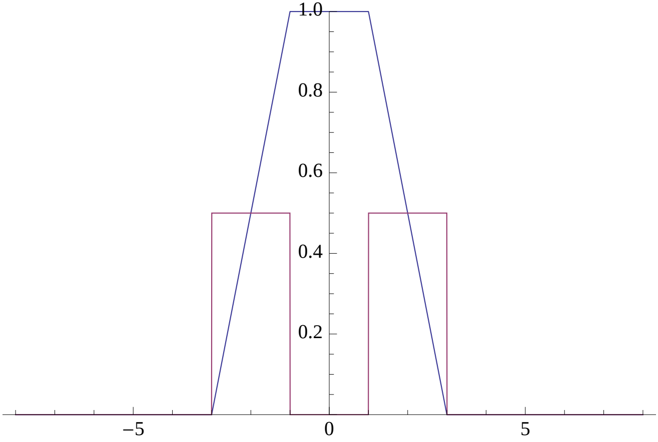
Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 1$:



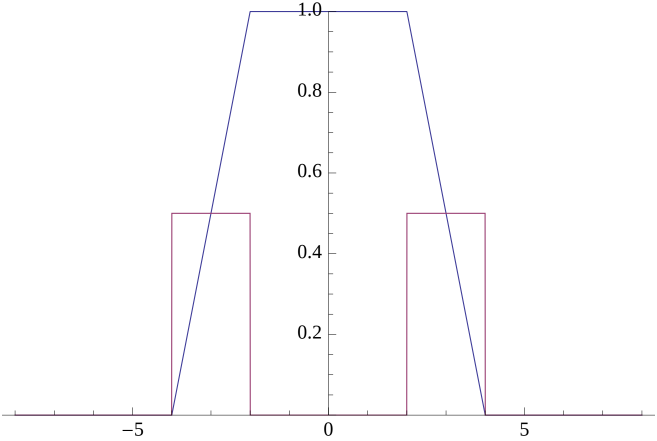
Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 2$:

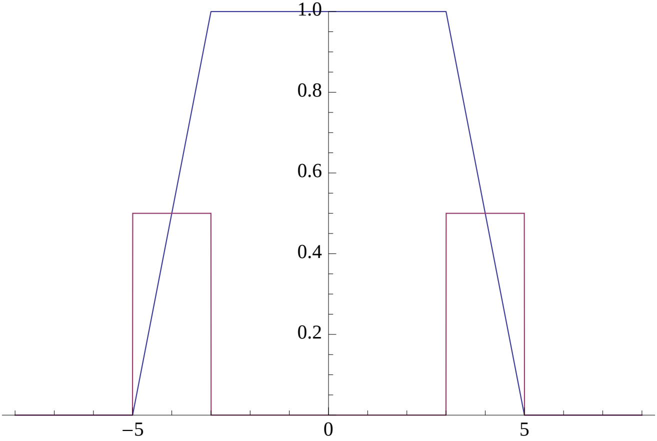


Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 3$:

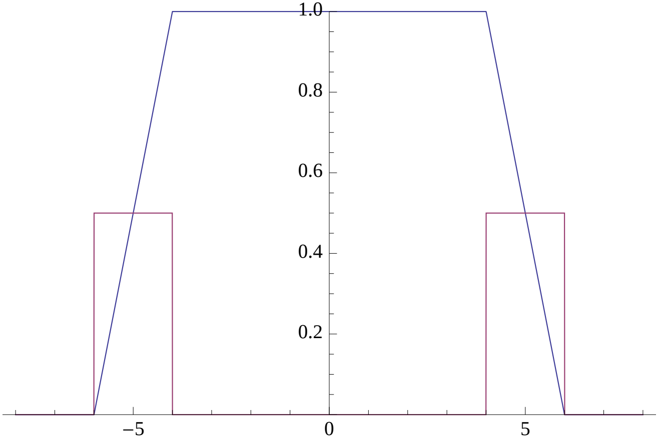


Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 4$:

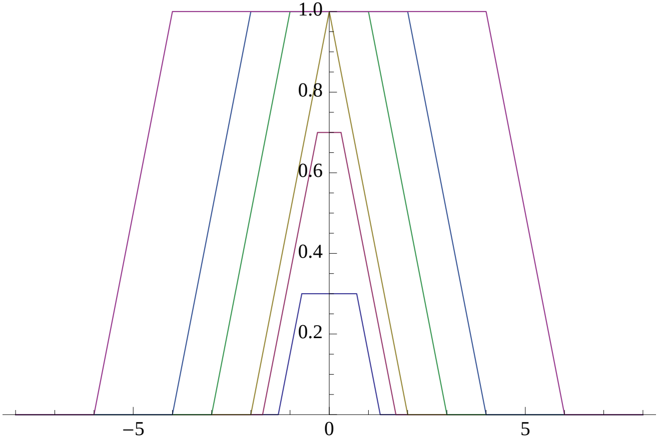


Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. Time $t = 5$:

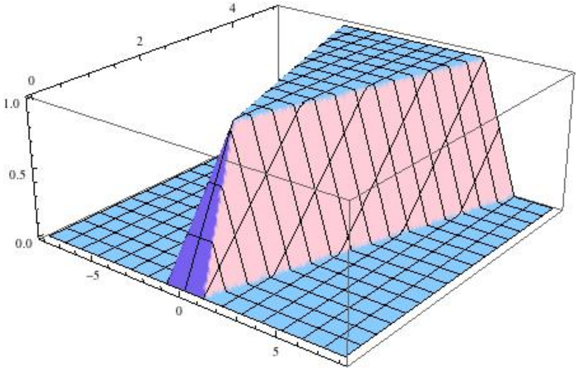


Initial conditions: $f(x)=0$ and $g(x)$ is a "rectangular impulse".
Speed $c=1$. Time $t=0.3, 0.7, 1, 2, 3, 5$:

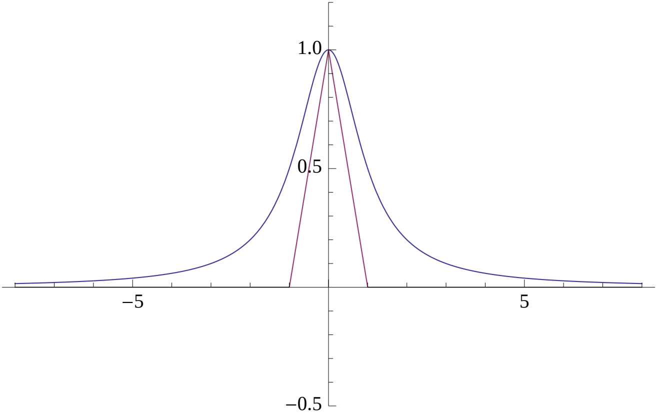


Hammer blow

Initial conditions: $f(x) = 0$ and $g(x)$ is a "rectangular impulse".
Speed $c = 1$. 3D plot:



Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. $t = 0$:



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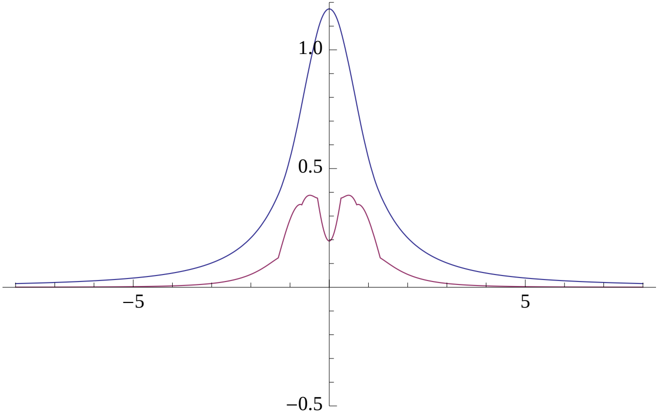
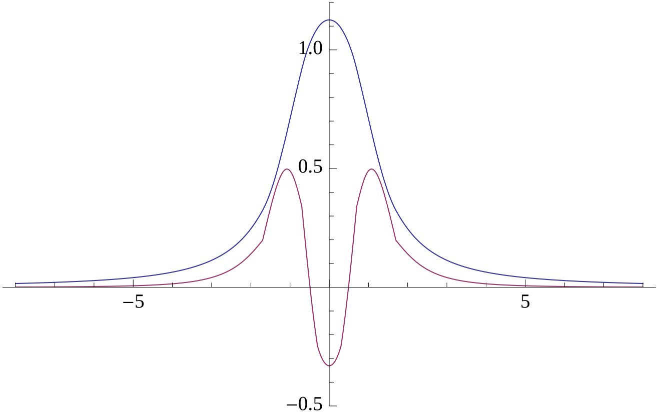
Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a “tent” function.
Speed $c = 1$. Time $t = 0.3$:



Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 0.7$:



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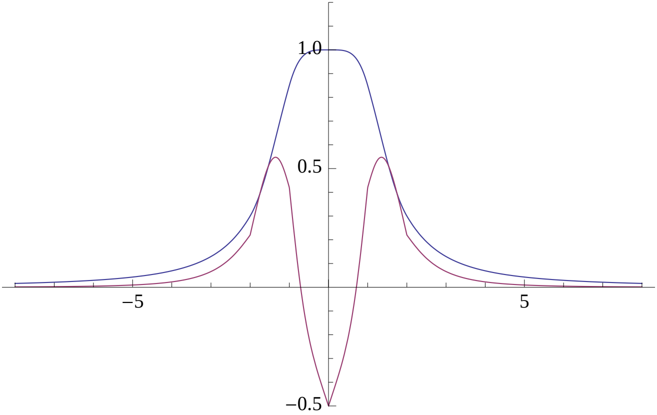
Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 1$:



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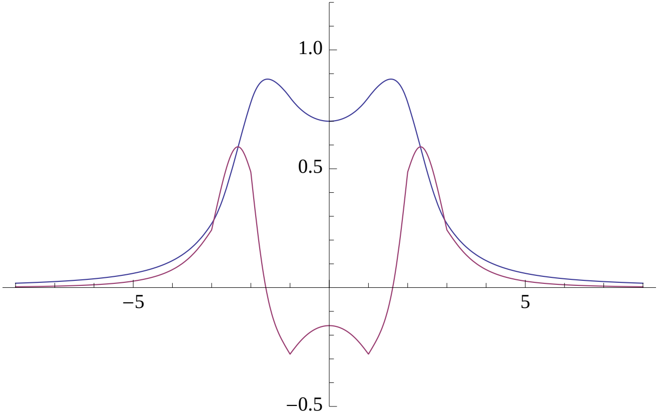
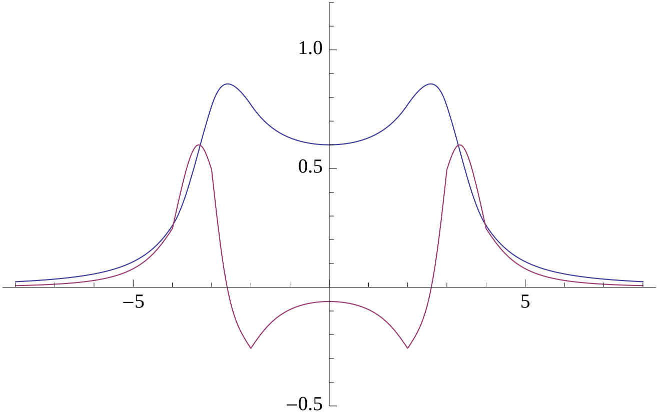
Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 2$:



Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 3$:



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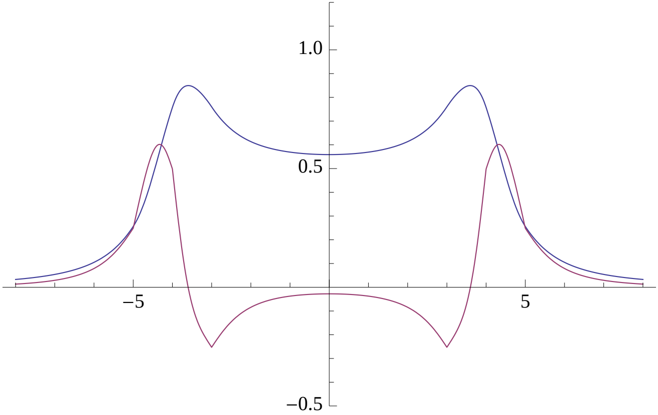
Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 4$:



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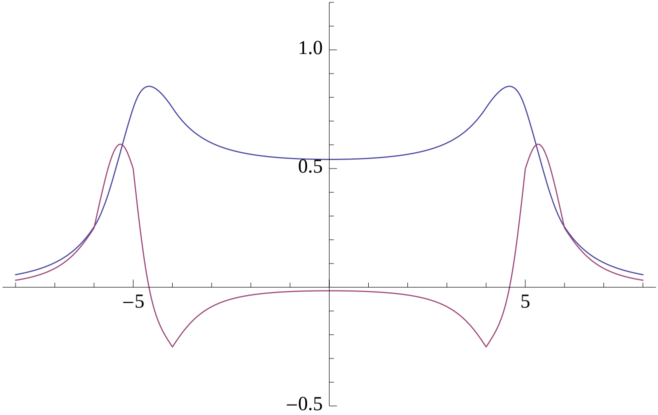
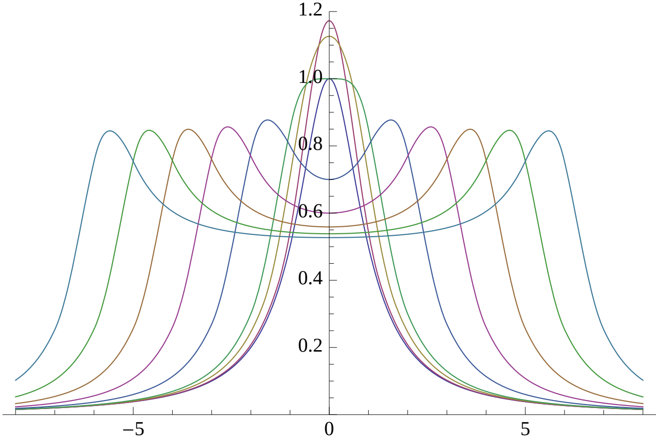
Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 5$:



Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. Time $t = 0.3, 0.7, 1, 2, 3, 4, 5, 6$:



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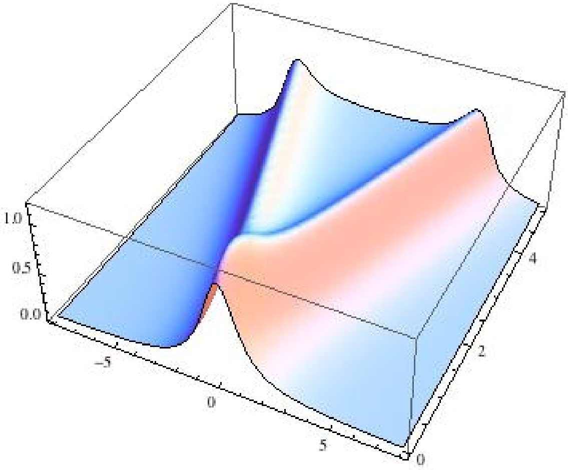
Initial conditions: $f(x) = \frac{1}{1+x^2}$ and $g(x)$ is a "tent" function.
Speed $c = 1$. 3D plot:



Finite speed of propagation

From D'Alambert's formula

$$u(x,t) = \tfrac{1}{2}f(x+ct) + \tfrac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s,$$

we see that $u(x,t)$ depends only on the initial data $f = u$ and $g = u_t$ on the interval $[x-ct, x+ct]$. This defines the **domain of dependence**.

Conversely, the value of u and u_t at (x_0, t_0) affects the solution at (x,t) only if $x_0 - c(t-t_0) \le x \le x_0 + c(t-t_0)$. This defines the **range of influence** of the point (x_0, t_0).

In particular, speed of propagation of information does not exceed c.