Lecture 10: Wave equation in 1D

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Math 319: Introduction to PDEs

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Wave equation

$$u_{tt} - c^2 u_{xx} = 0.$$

Let us introduce new variables $v_1 = u_t$ and $v_2 = cu_x$. Then we have

$$(v_1)_t - c(v_2)_x = 0, \qquad (v_2)_t - c(v_1)_x = 0,$$

or

$$v_t + Av_x = 0, \qquad \text{where} \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \text{and} \qquad A = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}.$$

We have

$$A = PDP^T \qquad \text{with} \qquad D = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}, \qquad \text{and} \qquad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This reveals that there are two characteristic velocities, c and $-c$.

Let us introduce the **characteristic variables** $\xi = x - ct$ and $\eta = x + ct$. We have

$$u_t = cu_\eta - cu_\xi, \qquad u_{tt} = c^2 u_{\eta\eta} - c^2 u_{\eta\xi} - c^2 u_{\xi\eta} + c^2 u_{\xi\xi},$$

and

$$u_x = u_\eta + u_\xi, \qquad u_{xx} = u_{\eta\eta} + u_{\eta\xi} + u_{\xi\eta} + u_{\xi\xi}$$

Therefore

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\eta\xi} = 0.$$

We have $(u_\eta)_\xi = 0$, so u_η does not depend on ξ, meaning that $u_\eta(\xi,\eta) = f(\eta)$ for some function f. Integrating over η, we get

$$u(\xi,\eta) = F(\eta) + G(\xi),$$

or in the original coordinates,

$$u(x,t) = F(x+ct) + G(x-ct).$$

D'Alambert's solution

The *initial value problem for the wave equation* is

$$u_{tt} - c^2 u_{xx} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

We know u must be of the form

$$u(x,t) = F(x+ct) + G(x-ct),$$

hence, putting $t = 0$, we have

$$F(x) + G(x) = f(x).$$

To use the other initial condition, let us find the t-derivative of u

$$u_t(x,t) = cF'(x+ct) - cG'(x-ct).$$

Now for $t = 0$ we have

$$cF'(x) - cG'(x) = g(x), \qquad \text{or} \qquad F(x) - F(0) - G(x) + G(0) = \frac{1}{c}\int_0^x g(s)\mathrm{d}s.$$

We derived

$$F(x)+G(x)=f(x), \qquad F(x)-F(0)-G(x)+G(0)=\frac{1}{c}\int_0^x g(s)\mathrm{d}s.$$

From this, we find

$$F(x)=\frac{f(x)+F(0)-G(0)}{2}+\frac{1}{2c}\int_0^x g(s)\mathrm{d}s,$$

and

$$G(x)=\frac{f(x)-F(0)+G(0)}{2}-\frac{1}{2c}\int_0^x g(s)\mathrm{d}s,$$

Finally, putting these two together, we get **D'Alambert's formula**

$$u(x,t)=F(x+ct)+G(x-ct)=\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_0^{x+ct} g(s)\mathrm{d}s$$

$$-\frac{1}{2c}\int_0^{x-ct} g(s)\mathrm{d}s=\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s$$

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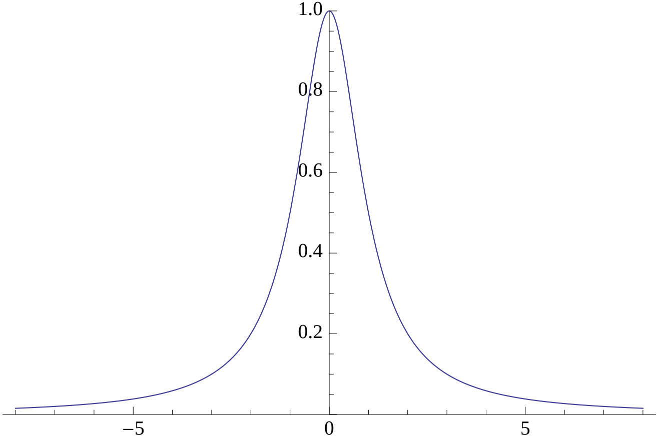
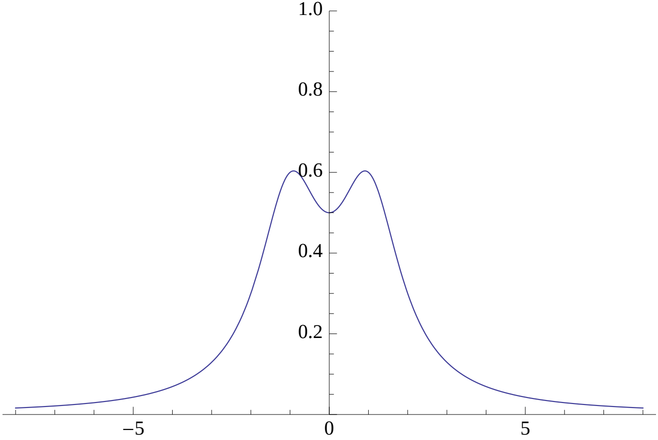
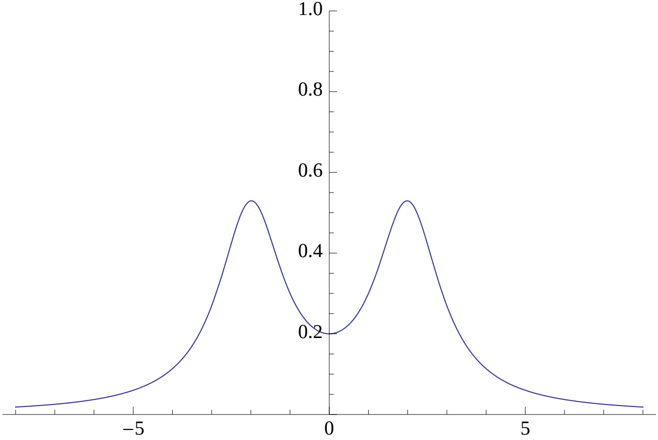
Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 0$:



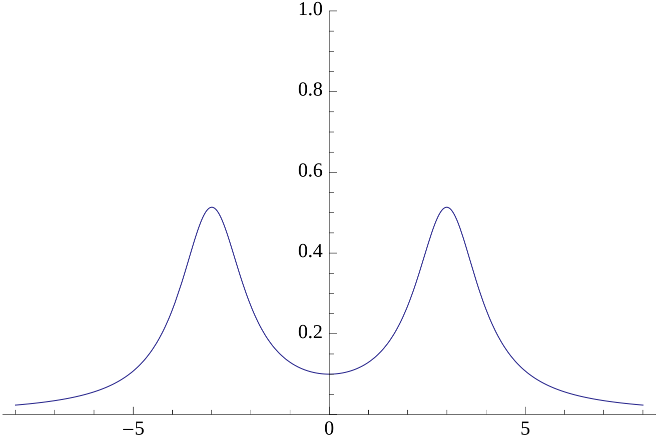
Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 1$:



Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 2$:



Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 3$:



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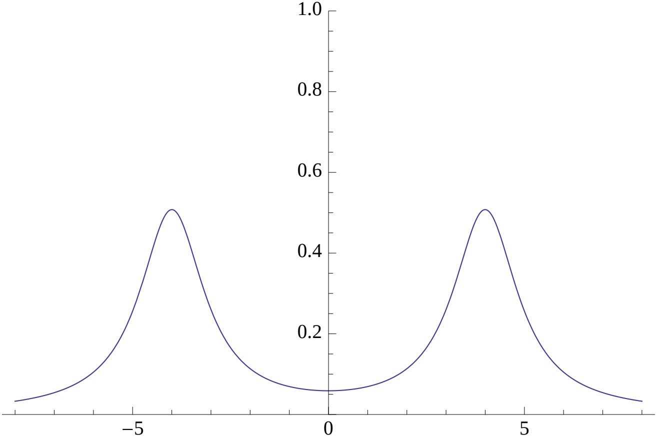
Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 4$:



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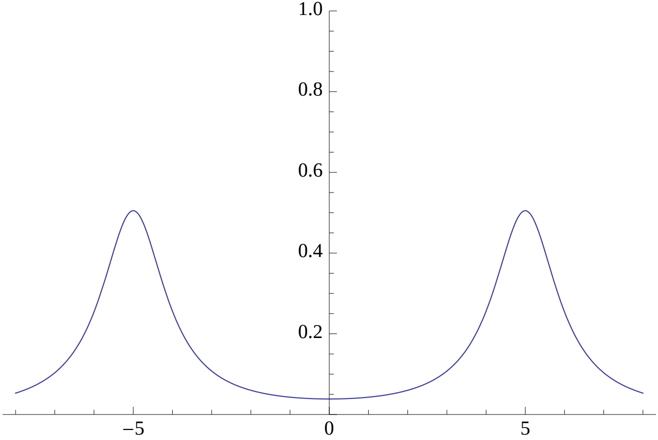
Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. Time $t = 5$:



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Initial conditions: $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = 0$. Speed $c = 1$. 3D plot: