Lecture 1: Basic concepts

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Math 319: Introduction to partial differential equations
McGill University, Montréal

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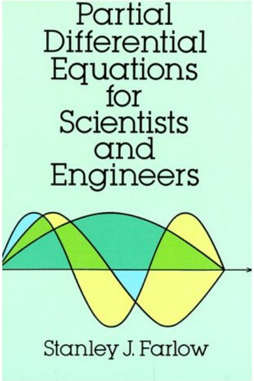
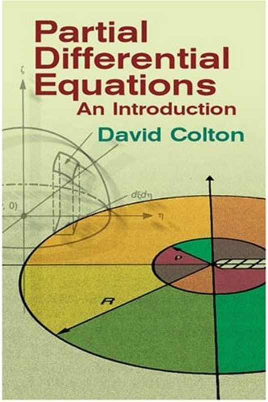
Organization of the course

- MyCourses and www.math.mcgill.ca/gantumur/math319
- Biweekly homework (posted on MyCourses) 20%
- Take-home midterm 30%
- Final exam ("cheatsheet" allowed) 50%
- Lectures: MTR 11:35-12:25, Burnside Hall 1B39
- Office hours: Thursdays 14:00-16:00, Burnside Hall 1123

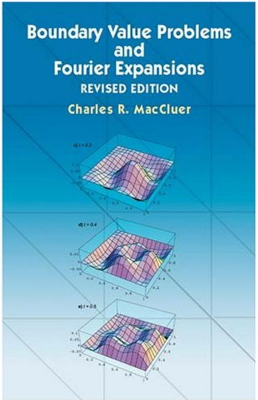
Textbooks

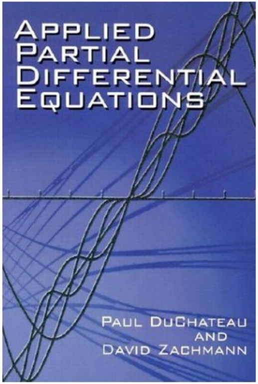




- *Also*: J. D. Logan. Applied partial differential equations

Textbooks





- *Also*: E. C. Zachmanoglou and D. W. Thoe. Introduction to partial differential equations with applications

What is a partial differential equation (PDE)?

Suppose that u is an unknown function of two variables x and a.
Partial derivatives of u are

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_a = \frac{\partial u}{\partial a}, \quad u_{xa} = \frac{\partial^2 u}{\partial x \partial a}, \quad \text{etc.}$$

A **differential equation** for the unknown function u is an equation involving u, its (partial) derivatives, and x and a.

Partial differential equations: $u_x + x^2 u_a = u$, $u_{xx} - u_{aa} = 0$, $u_{xx} + u_a u_x = 5a$

Ordinary differential equations: $u_x = u^3$, $u_a - u_{aa} = x$, $u_x - u_{xx} = \sqrt{u} + a^2$

Algebraic equations: $u^2 + u = x + a$, $u^5 + u^2 = x$, $\sin u + \cos u = au$

What is a function?

Let D be a subset of $\mathbb{R}^2$. For example,

- $D = [0,1]^2$ the unit square
- $D = \{(x,a) : x^2 + a^2 < 1\}$ the unit disk
- $D = \{(x,a) : a > 0\}$ the upper half plane
- $D = \{(x,a) : x > 0, a > 0\}$ the first quadrant

A **function** with domain D is a rule that assigns a number to each point in D. We write $u : D \to \mathbb{R}$ to indicate that u is a function with domain D and that it takes values from $\mathbb{R}$.

Examples:

- $u(x,a) = x^3 + a^2$ on the half plane $\{a > 0\}$
- $u(x,a) = \dfrac{1}{x-a}$ on $D = \{(x,a) : x \neq a\}$
- On the unit circle, $u(x,a) = \begin{cases} 1, & \text{if } x + a \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

Continuity and differentiability

u is **continuous** at (x, a) if $u(z, b) \to u(x, a)$ as $(z, b) \to (x, a)$.

If u is continuous at each $(x, a) \in D$, then we say u is continuous on D.

$u_x(x, a) = \lambda \in \mathbb{R}$ means $\dfrac{u(x+t, a) - u(x, a)}{t} \to \lambda$ as $t \to 0$. We say then u is **differentiable** at (x, a) **along** (or with respect to) x.

If $u_x(x, a)$ exists for all $(x, a) \in D$, then $u_x : D \to \mathbb{R}$ is a function with domain D.

If u_x and u_a both exist and are continuous on D, then we say u is **continuously differentiable** on D.

Directional derivatives

Let $n = (n_x, n_a)$ be a vector in $\mathbb{R}^2$.

If

$$\frac{u(x + t\,n_x, a + t\,n_a) - u(x, a)}{t} \to \lambda, \qquad \text{as} \quad t \to 0,$$

for some number $\lambda \in \mathbb{R}$, then we say u is **differentiable** at (x, a) **along** n, and write $u_n(x, a) = \lambda$.

We see that $u_x = u_n$ with $n = (1, 0)$, and $u_a = u_n$ with $n = (0, 1)$.

If u is **continuously differentiable** on D, then

$$u_n = n \cdot \nabla u \equiv n \cdot (u_x, u_a) \equiv n_x u_x + n_a u_a.$$

The vector-valued function $\nabla u : D \to \mathbb{R}^2$ is called the *gradient* of u.

What is a solution?

With a PDE, usually comes a domain D on which its solutions are supposed to live. A function $u : D \to \mathbb{R}$ is a **solution** to a PDE if u and its derivatives satisfy the PDE.

For example, $u(x,a) = x + a$ is a solution to $u_x - u_a = 0$. So is $u(x,a) = x + a + 1$. PDEs typically have plenty of solutions.

In fact, pick any differentiable function $f : \mathbb{R} \to \mathbb{R}$. Then $u(x,a) = f(x+a)$ is a solution to $u_x - u_a = 0$. One can think of PDEs as imposing some restrictions on the allowed behaviour of functions, and usually this restriction is not enough to pin down exactly one function.

Further insight is gained by noting that $u_x - u_a \equiv u_n$ with $n = (1,-1)$. So $u_x - u_a = 0$ is the same as $u_n = 0$, and any function that is constant along the direction n satisfies our equation. The same reasoning applies to, e.g., the equation $u_x + u_a = 0$.