MATH 319 MIDTERM EXAM

DUE THURSDAY MARCH 10

Note: All questions are of equal weight.

Analytic exercises

1. Find $u(x,t)$ satisfying
$$u_x + tu_t = -u, \qquad \text{and} \qquad u(x,1) = f(x), \tag{1}$$
where f is a continuously differentiable function. What are the characteristics? Plot the solution for different values of t.
2. Solve the wave equation
$$u_{tt} - u_{xx} = 0, \tag{2}$$
on the half-line $x \in (0,\infty)$ and for $t \in [0,\infty)$, with the boundary condition $u(0,t) = 0$ for $t > 0$, and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ for $x > 0$. The initial conditions must satisfy the consistency conditions $f(0) = 0$ and $g(0) = 0$. This is a model of vibration of a long elastic string with one end fixed. One way to solve the problem is to carry out the following program.
   - Extend the initial conditions f and g to the whole real line $\mathbb{R}$ in such a way that $f(-x) = -f(x)$ and $g(-x) = -g(x)$.
   - Solve the wave equation for u with the above extended inital data on the whole line by using D'Alambert's formula.
   - Check that the obtained solution satisfies $u(0,t) = 0$.
   - Take the part of $u(x,t)$ that is in the half-line $x > 0$ as the solution to the original problem.

   Note that this method is similar in spirit to the method of image charges. Now take $g = 0$, and take f to be highly localized (such as a bump, or a tent function), and plot time snapshots of the solution at several different time moments to study the behavior of propagation of the initial bump. When a wave reaches the left boundary $x = 0$, does it get reflected? Comment on the result, keeping in mind that the solution models the vibration of an elastic string.
3. The same as the preceding problem, but now the boundary condition $u(0,t) = 0$ is replaced by $u_x(0,t) = 0$. The initial conditions must satisfy the consistency conditions $f_x(0) = 0$ and $g_x(0) = 0$. This is a model of vibration of a long elastic string with one end attached to a bead that is able to move freely in the direction orthogonal to the string. Repeat the steps of the preceding problem, but note that the same extension of

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the initial data may not work. In particular, what are the similarities and differences between this and the preceding problem?

4. Now we solve the heat equation
$$u_t - u_{xx} = 0, \tag{3}$$
on the half-line $x \in (0, \infty)$ and for $t \in [0, \infty)$, with the boundary condition $u(0, t) = 0$ for $t > 0$, and initial condition $u(x, 0) = f(x)$ for $x > 0$. The initial condition must satisfy the consistency condition $f(0) = 0$. This is a model of heat conduction in a long pipe with the temperature at one end held fixed. One can solve this problem (as in Exercise 3) by extending the initial condition to $\mathbb{R}$ as an odd function, solving the problem by the solution formula for the heat equation on $\mathbb{R}$ involving the fundamental solution, and finally restricting attention to $x > 0$. Similarly to what was asked in Exercise 3, study the behavior of the solution for the initial condition $f(x) = \delta(x - 1)$ for $x > 0$, and comment on the physical interpretation of the result. Recall that the fundamental solution of the heat equation is just the solution with the initial condition given by the delta function $\delta(x)$, and that the initial condition $u(x, 0) = \delta(x - 1)$ represents unit amount of heat injected at point $x = 1$ when clock starts ticking.
5. The same as the preceding problem, but now the boundary condition $u(0, t) = 0$ is replaced by $u_x(0, t) = 0$. The initial condition must satisfy the consistency condition $f_x(0) = 0$. This is a model of heat conduction in a long pipe with one end completely insulated. Repeat the steps of the preceding problem, keeping in mind that you might have to come up with a different way to extend the initial data. In particular, what are the similarities and differences between this and the preceding problem?

Computational exercises

6. In this exercise, we will numerically solve the wave equation
$$u_{tt} - u_{xx} = 0, \tag{4}$$
on the interval $(0, 1)$, with the boundary conditions $u(0, t) = u_x(1, t) = 0$ for $t > 0$, and the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$ for $x \in (0, 1)$. Implement a simple finite difference scheme for solving the problem, or use the Matlab code `wave.m` from the course web page. The latter code is based on the scheme described in Chapter 10.4 of Peter Olver's online book. In all simulations, employ some highly localized smooth initial data f, such as the one given in `smoothbump.m`. Solve the problem with different spatial mesh-sizes, and different time-steps to illustrate the CFL condition, i.e., that too large time-steps lead to instability, with the threshold value given by the CFL condition. In the stable regime, compute the solution and plot time snapshots at several different time moments, and comment on the result. Especially observe how the waves reflect at the boundaries and what happens when they meet again.
7. The same as the preceding problem, but we replace the right boundary condition $u_x(1, t) = 0$ by $u(1, t) = 0$. Modify the code to accomodate this change. Repeat the steps of the preceding problem.