MATH 319 ASSIGNMENT 4

DUE THURSDAY APRIL 7

1. Let $D = \{(r, \phi) : 0 < r < 1, 0 < \phi < \pi\}$ be a half disk, where (r, ϕ) are the polar coordinates. Solve the initial value problem for the wave equation $u_{tt} = \Delta u$ on D with the homogeneous Dirichlet condition on the curved part $\{(1, \phi) : 0 < \phi < \pi\}$, and the homogeneous Neumann condition on the straight part $\{(r, \phi) : 0 \leq r < 1; \phi = 0, \pi\}$ of the boundary.
2. Show that the function
$$y(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin\theta) \mathrm{d}\theta,$$
satisfies the Bessel equation
$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$
This gives an integral representation for the Bessel function J_n. Show that $|J_n(x)| \leq 1$ for all $x \in \mathbb{R}$.
3. (J. Goldstein) Solve the heat equation $u_t = k\Delta u$ in the ball of radius R, with $u = \beta$ on the boundary and $u = \alpha$ when $t = 0$, where α and β are constants. Now consider the model of an egg as a homogeneous ball of radius π. Initially at 20°C, it is placed in a pot of boiling water at 100°C. How long does it take for the center to reach 60°C? Assume the heat conductivity $k = 0.006\,\mathrm{cm}^2/\mathrm{sec}$. (*Hint*: The problem is radial. Use the first term of the expansion to approximate the temperature at the center.)
4. Solve the Dirichlet boundary value problem for the Laplace equation $\Delta u = 0$ in the *exterior* of the 3-dimensional unit sphere. Allow only solutions that are bounded at inifinity. What simplifications do we get in the solution process if the Dirichlet boundary datum is independent of the azimuthal angle (longitude) φ?
5. Consider a bounded domain, and assume that the homogeneous Neumann boundary condition is imposed on all problems we consider in this question. Show that the Laplacian posseses an eigenvalue equal to zero, i.e., show that there is a nonzero function v (having the homogeneous Neumann boundary condition) such that $\Delta v = 0$. What would be the effect of this zero eigenvalue on the solutions of the heat equation $u_t = \Delta u$, and of the wave equation $u_{tt} = \Delta u$? What can you say about the solvability of the Poisson equation $\Delta u = f$?

Date: Winter 2011.