MATH 319 ASSIGNMENT 3

DUE THURSDAY MARCH 24

1. Use separation of variables to solve the heat conduction problem
$$u_t = u_{xx}, \qquad u(x,0) = f(x),$$
on the interval $(0,1)$, with the boundary conditions $u(0,t) = \alpha$ and $u_x(1,t) = \beta$ for $t > 0$, where α and β are given constants. Give an explicit expression for the solution in the case $f(x) = \alpha + \frac{\beta}{2}x^2$.
2. Use separation of variables to solve
$$u_{tt} = u_{xx} - u, \qquad u(x,0) = 0, \qquad u_t(x,0) = 1 + \cos^3 x,$$
on the interval $(0,\pi)$, with the homogeneous Dirichlet boundary conditions.
3. Let $\alpha \in (0, 2\pi)$ be given, and let $W_\alpha = \{(r,\phi) : 0 < r < 1,\, 0 < \phi < \alpha\}$ be a pie shaped domain with the central angle α, where (r,ϕ) are the polar coordinates. Use separation of variables to solve the Laplace problem
$$\Delta u = 0,$$
on W_α, with the boundary conditions
$$u(r,0) = u(r,\alpha) = 0, \quad \text{for} \quad r \in [0,1), \qquad \text{and} \qquad u(1,\phi) = f(\phi) \quad \text{for} \quad \phi \in (0,\alpha),$$
where f is a given function. Give an explicit expression for u when $f(\phi) = \sin\left(\frac{\pi}{\alpha}\phi\right)$.
4. Let $\mathbb{D} = \{(x,y) : x^2 + y^2 < 1\}$ be the unit disk. Show that
$$\Delta u = u^3 \quad \text{in} \quad \mathbb{D}, \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\mathbb{D},$$
has no twice continuously differentiable solution other than $u \equiv 0$. (*Hint*: Δu is non-positive at the maximum of u. Another approach is to multiply the both sides of the equation by u and integrate.)
5. Let Ω be a 3-dimensional bounded domain with smooth boundary. Show that the eigenvalues of the Laplacian on Ω with homogeneous Neumann boundary condition cannot be positive. Show also that the eigenvalues of the Laplacian on Ω with homogeneous Dirichlet boundary condition are strictly negative. (*Hint* for the Dirichlet case: The Laplace equation has a unique solution with the homogeneous Dirichlet boundary condition.)

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