MATH 319 ASSIGNMENT 2

DUE THURSDAY FEBRUARY 17

1. Find $u(x, a)$ satisfying

$$xu_x + au_a = 0, \qquad \text{and} \qquad u(x, 1) = f(x),$$

where f is a continuously differentiable function.

2. Find the general solution of
   a) $u_{xx} + 6u_{xy} - 16u_{yy} = 0$,
   b) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$.
3. Reduce the following equations to canonical form:
   a) $a^2u_{xx} + 2xau_{xa} + x^2u_{aa} = 0$,
   b) $u_{xx} - 2xu_{xa} = 0$.
4. Derive the analogue of D'Alambert's formula for the solution of the Cauchy problem

$$u_{tt} - u_{xx} = \lambda u, \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = g(x),$$

where λ is a constant, and f and g are given functions.

5. Show that the general solution of the hyperbolic equation

$$u_{xx} - tu_{tt} - \tfrac{1}{2}u_t = 0, \qquad (t > 0),$$

has the form

$$u(x, t) = F(x + 2\sqrt{t}) + G(x - 2\sqrt{t}).$$

6. Let $u(x, y, t)$ be the solution of the Cauchy problem

$$u_{tt} - c^2u_{xx} - c^2u_{yy} = 0, \qquad u(x, y, 0) = f(x, y), \qquad u_t(x, y, 0) = g(x, y),$$

where $c > 0$ is a constant, and $f(x, y)$ and $g(x, y)$ vanish for $x^2 + y^2 > r^2$ for some $r > 0$. Show that the solution $u(x, y, t)$ vanishes if $x^2 + y^2 - r^2 > c^2t^2$.

7. Find all solutions of the heat equation $u_t = u_{xx}$ of the form

$$u(x, t) = \frac{1}{\sqrt{t}} v(\frac{x}{\sqrt{t}}).$$

8. Show that if $u(x, t)$ is a solution of $u_t = u_{xx}$, then so is

$$v(x, t) = t^{-1/2}e^{-x^2/(4t)}u(x/t, -1/t).$$

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9. Let $u(x,t)$ satisfy the heat equation for $x \in (0,1)$ and $t > 0$, the boundary conditions $u(0,t) = u(1,t) = 0$ for $t \geq 0$, and the initial condition $u(x,0) = f(x)$ for $x \in [0,1]$ with f a continuously differentiable function. Show that

$$\int_0^1 |u(x,t)|^2 \mathrm{d}x \leq \int_0^1 |f(x)|^2 \mathrm{d}x, \qquad \text{for any} \quad t \geq 0.$$

(*Hint*: Use $2uu_t = (u^2)_t$ and $uu_{xx} = (uu_x)_x - (u_x)^2$.) Derive a uniqueness theorem for the above initial-boundary value problem.