MATH 319 ASSIGNMENT 1

DUE THURSDAY JANUARY 27

1. Find all solutions $u(x,a)$ of the problem

$$3u_x + u_a = 0, \qquad \text{satisfying} \qquad u(x,0) = e^{-x^2}\cos x.$$

   Sketch $u(x,a)$ as a function of x for a few fixed values of a, e.g., for $a = 0, 1, 2, 3$.
2. Which of the following operators L are linear? Justify your answer.
   a) $Lu = x^2 u_{xx} + u_a \sin x$
   b) $Lu = (u_x)^3 - u_a \sin x$
   c) $Lu = u^2 u_x - u_a \sin x$
   d) $Lu = a^2 u_x - e^x u_a + axu$
   e) $Lu = (e^x u_x + u)_x$
3. Consider the Poisson equation $u_{xx} = f$ on the interval $(0,1)$ subject to the boundary conditions $u_x(0) = \alpha$ (Neumann), and $u(1) = \beta$ (Dirichlet). Find the solution using Green's function approach.
4. Consider the steady-state *convection-diffusion* (or *advection-diffusion*) problem

$$-\kappa v_{xx}(x) + a v_x(x) = f(x), \qquad x \in (0,L); \qquad v(0) = \alpha, \quad v(L) = \beta, \tag{1}$$

   where α, β, $\kappa > 0$, $a > 0$, and $L > 0$ are constants, and f is a given function. This problem models the temperature distribution of a fluid flowing through a pipe with constant velocity a. The fluid has constant heat diffusion coefficient κ, and f represents heat sources distributed along the walls of the pipe. Moreover, the constant temperatures α and β are maintained at both ends of the pipe, and the pipe length is L. The above problem is interesting when the *Péclet number* $Pe = aL/\kappa$ is large, i.e., when convection dominates diffusion. In this regime, we expect a sharp *boundary layer* forming near the right (called *outflow*) boundary.
   a) The quantities in (1) have physical dimensions; for instance v has a dimension of temperature, and L has a dimension of length. We shall now "nondimensionalize" the problem. By defining $u(x) = \lambda v(xL)$ for a suitable $\lambda > 0$, and also suitably scaling the data f, α and β, transform (1) to the form

$$-\varepsilon u_{xx}(x) + u_x(x) = f(x), \qquad x \in (0,1); \qquad u(0) = \alpha, \quad u(1) = \beta. \tag{2}$$

      Express ε in terms of the (original) dimensional data. (The case with $\varepsilon \neq 0$ is called a *singular perturbation* of the case with $\varepsilon = 0$, because (2) is a second order equation as long as ε is nonzero no matter how small.)

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b) Find the exact solution of the following problem

$$-\varepsilon u_{xx}(x) + u_x(x) = 1, \qquad x \in (0,1); \qquad u(0) = u(1) = 0. \tag{3}$$

Plot the exact solution for different values of ε (e.g., $\varepsilon = 1, \frac{1}{10}, \frac{1}{50}, \frac{1}{100}$) and comment on the behaviour as $\varepsilon \to 0$.

c) When we set $\varepsilon = 0$ in (3), the differential equation becomes of first order, and we cannot specify two boundary conditions at 0 and 1 simultaneously. Thus we guess that in the limit $\varepsilon \to 0$, the solution of (3) will resemble the solution of the first order differential equation that results from setting $\varepsilon = 0$ in (3) and omitting one of the boundary conditions. Identify with justification, which of the boundary conditions we should omit.

5. Derive the expression for the Laplacian $\Delta u = u_{xx} + u_{yy}$ in polar coordinates. Find all radially symmetric solutions of the Laplace equation $\Delta u = 0$ for $r > 0$ (meaning that $\Delta u = 0$ holds except possibly at the origin).
6. Prove that $\nabla \times \nabla u = 0$ (i.e., $\operatorname{curl} \operatorname{grad} u = 0$) and $\nabla \cdot (\nabla \times A) = 0$ (i.e., $\operatorname{div} \operatorname{curl} A = 0$), where u and A are respectively scalar and vector fields in 3 dimensions.
7. Which ones are harmonic functions? Justify your answer.
   a) $u(x,y) = x^2 + y^2$
   b) $u(x,y) = x^2 - y^2$
   c) $u(x,y) = e^x \sin y$
   d) $u(x,y) = \log(x^2 + y^2)$
   e) $u(x,y) = 3x^2y - y^3$
8. A general bivariate polynomial of degree less or equal to 3 is

$$p(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}.$$

   Find all such polynomials that are harmonic (p is harmonic if $\Delta p = 0$ everywhere).
9. A cavity in a conductor has the shape of a half sphere, being bounded by the surfaces $\theta = 0$ and $r = R$ in spherical coordinates. A point charge q is located at $\theta = \theta_q$, $r = r_q$, and $\phi = 0$, where $0 < \theta_q \leq \frac{\pi}{2}$ and $0 < r_q < R$.
   a) Give the locations and magnitudes of the image charges necessary to maintain the electric potential $\varphi = 0$ at the boundary of the cavity.
   b) Write down the potential inside the cavity.