POWER SERIES SOLUTIONS

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Abstract. We introduce the power series solution method for second order linear differential equations, and illustrate it by examples.

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1. Ordinary and singular points

Consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{1}$$

where $a(x)$, $b(x)$, and $c(x)$ are polynomials, or equivalently,

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where

$$p(x) = \frac{b(x)}{a(x)}, \qquad \text{and} \qquad q(x) = \frac{c(x)}{a(x)}. \tag{3}$$

Note that this makes $p(x)$ and $q(x)$ rational functions. We call (1) the *general form*, and (2) the *normal* or *standard form* of the same underlying equation.

We would like to look for a solution in the form of the power series

$$y(x) = \sum_{k=0}^{\infty} a_k (x-\alpha)^k, \tag{4}$$

which is centred at the point α. It turns out that the success of such a power series approach is strongly dependent on the behaviour of the coefficients $p(x)$ and $q(x)$ near the point α.

Definition 1. A point $\alpha \in \mathbb{C}$ in the complex plane is called an *ordinary point* of (2) if the functions $p(z)$ and $q(z)$ are continuous at $z = \alpha$, that is, if the limits

$$\lim_{z\to\alpha} p(z), \qquad \text{and} \qquad \lim_{z\to\alpha} q(z), \tag{5}$$

exist as finite complex numbers. If α is not ordinary, we call it a *singular point* of (2).

Remark 2. Since $p(x)$ and $q(x)$ are rational functions, there is no ambiguity in the meaning of $p(z)$ and $q(z)$ for complex variable z. For example,

$$\text{if} \qquad p(x) = \frac{2x^2+1}{3x^2-2x+5}, \qquad \text{then} \qquad p(z) = \frac{2z^2+1}{3z^2-2z+5}. \tag{6}$$

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Example 3. Consider the equation

$$(x^2-1)y''+(x^3-x^2)y'+(1-x)y=0. \tag{7}$$

Dividing the left hand side by x^2-1, we have

$$\frac{(x^2-1)y''+(x^3-x^2)y'+(1-x)y}{x^2-1}=y''+\frac{x^3-x^2}{x^2-1}y'-\frac{x-1}{x^2-1}y, \tag{8}$$

which, after some simplifications, leads us to the normal form

$$y''+xy'-\frac{1}{x+1}\,y=0. \tag{9}$$

Comparing this with (2), we identify the coefficients $p(x)$ and $q(x)$ as

$$p(x)=x, \qquad \text{and} \qquad q(x)=-\frac{1}{x+1}. \tag{10}$$

The function $p(z)=z$ is finite for all complex values of z, and $q(z)=-\frac{1}{z+1}$ is finite for all $z\in\mathbb{C}$ except at $z=-1$. In fact, we have $|q(z)|\to\infty$ as $z\to-1$. Therefore we conclude that the only singular point of the equation (7) is at $z=-1$, and all other points in the complex plane are ordinary points.

Example 4. Consider the equation

$$y''+\frac{y}{x^2-2x+2}=0. \tag{11}$$

We identify the coefficients $p(x)$ and $q(x)$ as

$$p(x)=0, \qquad \text{and} \qquad q(x)=\frac{1}{x^2-2x+2}. \tag{12}$$

Obviously, $p(x)$ is not going to play any role in determining ordinary and singular points of (11), so let us focus on the coefficient $q(x)$. The singularities of $q(z)$ are at the roots of the quadratic equation

$$z^2-2z+2=0, \tag{13}$$

which are

$$z_{1,2}=1\pm\sqrt{1-2}=1\pm i. \tag{14}$$

The function $q(z)$ is finite for all $z\in\mathbb{C}$ except at $z=1\pm i$, and we have $|q(z)|\to\infty$ as $z\to1\pm i$. Therefore we conclude that the only singular points of the equation (11) are at $z=1\pm i$, and all other points in the complex plane are ordinary points.

FIGURE 1. The singular points of the equations (7) and (11).

2. The fundamental theorem

We present the following theorem without proof, which forms the basis of power series solutions near an ordinary point.

Theorem 5. *If $\alpha \in \mathbb{R}$ is an ordinary point of (2), then there are two linearly independent solutions to (2), of the form*

$$y(x) = \sum_{k=0}^{\infty} a_k (x-\alpha)^k, \tag{15}$$

which converges (at least) in the interval $(\alpha - R, \alpha + R)$, where $R > 0$ is the distance from α to the nearest singular point of (2), in the complex plane.

Example 6. Consider the equation

$$(x^2-1)y'' + (x^3 - x^2)y' + (1-x)y = 0. \tag{16}$$

From Example 3, we know that its normal form is

$$y'' + xy' - \frac{1}{x+1}\, y = 0,. \tag{17}$$

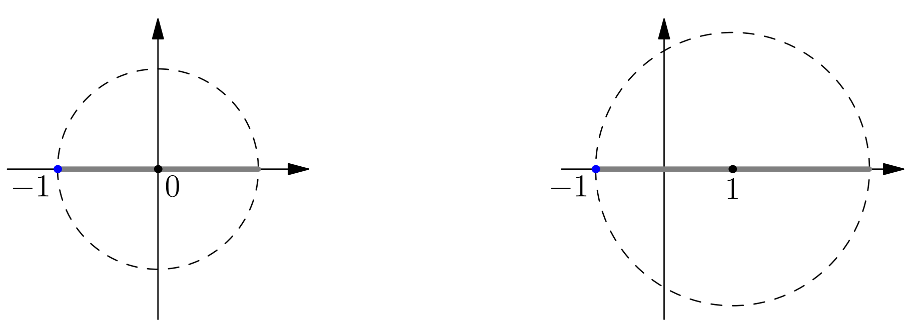
and that its only singular point is at $z = -1$.

Suppose that we want to find a solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{18}$$

that is, (15) with $\alpha = 0$. The distance between $\alpha = 0$ and the singular point at $z = -1$ is $R = 1$. Hence Theorem 5 guarantees that there are two linearly independent solutions of the form (23) that converges in the interval $(0-1, 0+1) = (-1, 1)$. Geometrically, this procedure of finding the convergence interval can be described as follows. We draw the largest possible circle centred at α (considered as a point on the real line), that does not have any singular point in its *interior*. Then the intersection of the circle's *interior* with the real line (i.e., the horizontal axis) gives the convergence interval. See Figure 2.

Now let us see how the convergence behaviour changes if we want to find a solution of the form (15) with $\alpha = 1$. In this case, we have $R = 2$, which is the distance between $\alpha = 1$ and the singularity at $z = -1$. So Theorem 5 guarantees convergence of the series (15) in the interval $(-1, 3)$.



(A) $\alpha = 0$ (B) $\alpha = 1$

Figure 2. The convergence intervals described in Example 6.

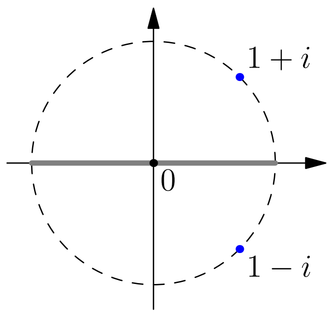
Example 7. Consider the equation

$$y'' + \frac{y}{x^2 - 2x + 2} = 0. \tag{19}$$

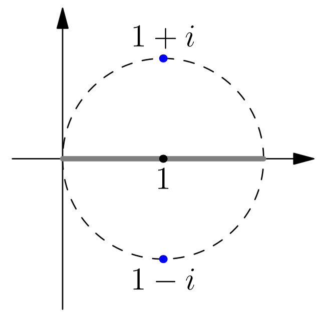
We have seen in Example 4 that the only singular points of this equation are $z = 1 \pm i$.

Suppose that we want to find a solution of the form (15) with $\alpha = 0$. The distance from $\alpha = 0$ to any of the singular points $z = 1 \pm i$ is $R = \sqrt{2}$, so Theorem 5 guarantees convergence of the series (15) in the interval $(-\sqrt{2}, \sqrt{2})$.

To take another random example, if $\alpha = 1$, then $R = 1$, hence the convergence interval given by Theorem 5 is $(0, 2)$.



(A) $\alpha = 0$



(B) $\alpha = 1$

FIGURE 3. The convergence intervals described in Example 7.

3. Formulation of the method

We shall now formulate a method to compute the coefficients a_k of the power series solutions (15) to the equation (1). For convenience, let us redisplay (1) here

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{20}$$

First of all, note that we can always assume $\alpha = 0$, which simplifies computations. Indeed, introducing the new variables $t = x - \alpha$ and $u(t) = y(t + \alpha)$, we have $u'(t) = y'(t + \alpha)$ and $u''(t) = y''(t + \alpha)$. Therefore (20) is equivalent to

$$a(t + \alpha)u'' + b(t + \alpha)u' + c(t + \alpha)u = 0, \tag{21}$$

and (15) becomes

$$u(t) = y(t + \alpha) = \sum_{k=0}^{\infty} a_k t^k. \tag{22}$$

This means that if we have a procedure to solve (20) in terms of the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{23}$$

then we will have a way to solve it in terms of the more general power series (15) with $\alpha \neq 0$, since we could just apply the same procedure to solve (21) in terms of (22).

For $\alpha = 0$, we formulate the power series solution method as follows.

(i) Compute $y'(x)$ and $y''(x)$ by differentiating (23) term by term, and substitute the resulting series, together with (23), into (20) to write it as

$$\sum_{k=0}^{\infty} f_k x^k = 0, \tag{24}$$

where the coefficients f_k depend on the unknown coefficients $a_0, a_1, \ldots$ of $y(x)$.

(ii) Impose the condition $f_k = 0$ for all $k = 0, 1, \ldots$, which give relations between the unknown coefficients $a_0, a_1, \ldots$ of $y(x)$.

(iii) Solve these relations to express the coefficients $a_2, a_3, a_4, \ldots$ in terms of a_0 and a_1. Since we are solving a second order equation, we expect there to be two arbitrary constants, and we are assigning these roles to a_0 and a_1.

Remark 8. Note that the method is quite general, but considerably less so than the method of integrating factors for first order linear equations

$$y' + P(x)y = 0. \tag{25}$$

Namely, while $P(x)$ in (25) can be almost any function, $a(x)$, $b(x)$ and $c(x)$ in (1) must be polynomials (which can be relaxed to analyticity).

4. Examples

In this section, we illustrate the power series method on concrete examples.

Example 9. Let us test the power series method on the simple equation

$$y' - 2y = 0. \tag{26}$$

Differentiating the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots, \tag{27}$$

term by term, we get

$$y'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots. \tag{28}$$

Writing (26) as $y' = 2y$, and comparing (28) with

$$2y(x) = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \ldots, \tag{29}$$

we derive the relations

$$a_1 = 2a_0, \qquad 2a_2 = 2a_1, \qquad 3a_3 = 2a_2, \qquad 4a_4 = 2a_3, \tag{30}$$

and so on, in general we have

$$(k+1)a_{k+1} = 2a_k, \qquad k = 0, 1, 2, \ldots. \tag{31}$$

Therefore we can express a_{k+1} in terms of a_k as

$$a_{k+1} = \frac{2}{k+1} a_k, \qquad k = 0, 1, \ldots. \tag{32}$$

Let us compute the first few coefficients explicitly:

$$a_1 = \frac{2}{1} a_0, \quad a_2 = \frac{2}{2} a_1 = \frac{2 \cdot 2}{2 \cdot 1} a_0, \quad a_3 = \frac{2}{3} a_2 = \frac{2 \cdot 2 \cdot 2}{3 \cdot 2 \cdot 1} a_0, \quad a_4 = \frac{2}{4} a_3 = \frac{2 \cdot 2 \cdot 2 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 1} a_0. \tag{33}$$

From here it is clear that

$$a_k = \frac{2^k}{k!} a_0, \tag{34}$$

and so

$$y(x) = \sum_{k=0}^{\infty} \frac{2^k}{k!} a_0 x^k = a_0 \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = a_0 e^{2x}. \tag{35}$$

We see that the coefficient a_0 is arbitrary, and all other coefficients a_k depend on this choice. This is of course consistent with the fact that the general solution of (26) must involve one arbitrary constant.

Example 10. We consider again the equation

$$y' - 2y = 0. \tag{36}$$

The problem is identical to the preceding example, but we will follow here a slightly different path to illustrate certain techniques to work with series. We differentiate

$$y(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{37}$$

under the sum sign, and get

$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}. \tag{38}$$

Then we notice that the term corresponding to the index $k = 0$ is zero, and so we can start the sum from the index $k = 1$, as

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}. \tag{39}$$

Finally, we introduce new summation variable $n = k - 1$, to write it as

$$y'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n. \tag{40}$$

Substituting (37) and (40) into the equation $y' - 2y = 0$, we get

$$\begin{aligned} 0 = y'(x) - 2y(x) &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2 \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - 2 \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} [(k+1) a_{k+1} - 2a_k] x^k, \end{aligned}$$

where in the second line we renamed the summation index in the first sum in order to conveniently combine the terms of the two sums. This leads to

$$(k+1) a_{k+1} - 2a_k = 0, \qquad k = 0, 1, \ldots, \tag{41}$$

and the rest of the solution is the same as in the preceding example.

Example 11. Consider the second order equation

$$y'' + y = 0, \tag{42}$$

and we look for a solution in the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{43}$$

Differentiating term by term, we get

$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \tag{44}$$

where we have introduced $n = k - 1$. Differentiating again, we have

$$y''(x) = \sum_{n=0}^{\infty} n(n+1) a_{n+1} x^{n-1} = \sum_{n=1}^{\infty} n(n+1) a_{n+1} x^{n-1} = \sum_{\ell=0}^{\infty} (\ell+1)(\ell+2) a_{\ell+2} x^{\ell}, \tag{45}$$

where we have introduced $\ell = n-1$. Substituting this into the equation $y'' + y = 0$, we get

$$\begin{aligned} 0 = y''(x) + y(x) &= \sum_{\ell=0}^{\infty} (\ell+1)(\ell+2)a_{\ell+2}x^{\ell} + \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}x^{k} + \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} [(k+1)(k+2)a_{k+2} + a_k]x^k, \end{aligned}$$

where in the second line we renamed the summation index in the first sum in order to conveniently combine the terms of the two sums. This leads to the relations

$$(k+1)(k+2)a_{k+2} + a_k = 0, \qquad k = 0, 1, \ldots, \tag{46}$$

that is,

$$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, \qquad k = 0, 1, \ldots. \tag{47}$$

Let us apply it for $k = 0, 2, 4$, and get

$$a_2 = -\frac{a_0}{2\cdot 1}, \qquad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}, \qquad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}. \tag{48}$$

From here it is clear that for even indices k, that is, for $k = 2n$ with $n = 0, 1, 2, \ldots$, we have

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0. \tag{49}$$

Similarly, applying (47) for $k = 1, 3, 5$, we infer

$$a_3 = -\frac{a_1}{3\cdot 2}, \qquad a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2}, \qquad a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}, \tag{50}$$

which implies that

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1, \qquad \text{for} \quad n = 0, 1, 2, \ldots. \tag{51}$$

We have two formulas (49) and (51) for the general coefficient a_k depending on the parity of the index k. In order to use these formulas in

$$y(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{52}$$

we first split the sum into two sums, so that one sum contains only even powers of x and the other sum contains only odd powers of x, as

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}. \tag{53}$$

Now we can apply (49) and (51), to get

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = a_0 \cos x + a_1 \sin x. \tag{54}$$

Example 12. Consider the problem of solving

$$y'' - xy = 0, \tag{55}$$

in terms of the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{56}$$

Recall from (45) that

$$y''(x) = \sum_{\ell=0}^{\infty} (\ell+1)(\ell+2)a_{\ell+2}x^{\ell} = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + 4\cdot 5a_5x^3 + \ldots. \tag{57}$$

On the other hand, we have

$$xy(x) = \sum_{k=0}^{\infty} a_kx^{k+1} = a_0x + a_1x^2 + a_2x^3 + \ldots. \tag{58}$$

Thinking of (55) as $y'' = xy$, and comparing (57) with (58), we get the relations

$$2a_2 = 0, \qquad 2\cdot 3a_3 = a_0, \qquad 3\cdot 4a_4 = a_1, \qquad 4\cdot 5a_5 = a_2, \tag{59}$$

which are, of course, but a few of infinitely many relations. Nevertheless, they make it clear that $a_2 = 0$, and that

$$(n-1)na_n = a_{n-3} \qquad \text{or} \qquad a_n = \frac{a_{n-3}}{n(n-1)}, \qquad \text{for} \quad n \geq 3. \tag{60}$$

In particular, we have $a_2 = a_5 = a_8 = \ldots = 0$, that is,

$$a_{2+3k} = 0, \qquad \text{for} \quad k = 0, 1, 2, \ldots. \tag{61}$$

To get some insight on the other coefficients, let us compute a few of them, as

$$\begin{aligned} a_3 &= \frac{a_0}{3\cdot 2}, & a_4 &= \frac{a_1}{4\cdot 3}, \\ a_6 &= \frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}, & a_7 &= \frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}, \\ a_9 &= \frac{a_6}{9\cdot 8} = \frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}, & a_{10} &= \frac{a_7}{10\cdot 9} = \frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}. \end{aligned}$$

So we have

$$a_{3k} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3k-1)\cdot(3k)} = a_0\frac{4\cdot 7\cdot 10\cdots(3k-2)}{(3k)!}, \tag{62}$$

for $k = 0, 1, 2, \ldots$, and

$$a_{3k+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3k)\cdot(3k+1)} = a_1\frac{2\cdot 5\cdot 8\cdots(3k-1)}{(3k+1)!}, \tag{63}$$

for $k = 0, 1, 2, \ldots$. We can write the final solution as

$$y(x) = a_0\sum_{k=0}^{\infty}\frac{4\cdot 7\cdots(3k-2)}{(3k)!}x^{3k} + a_1\sum_{k=0}^{\infty}\frac{2\cdot 5\cdots(3k-1)}{(3k+1)!}x^{3k+1}, \tag{64}$$

which shows that we have 2 arbitrary constants. Let $y_0(x)$ be given by $y(x)$ in (64) with $a_0 = 1$ and $a_1 = 0$, and similarly, let $y_1(x)$ be given by $y(x)$ with $a_0 = 0$ and $a_1 = 1$. Then $y_0(x)$ and $y_1(x)$ are linearly independent, because $y_0(0) = 1$ and $y_1(0) = 0$.

Exercise 13. Find two linearly independent solutions of

$$y'' + x^2y' = 0, \tag{65}$$

in terms of the power series

$$y(x) = \sum_{k=0}^{\infty} a_kx^k. \tag{66}$$