Math 315 Midterm Exam Winter 2014

Version 4

1. Solve $x^2 e^{x^2+y}(2x^2+3)\,\mathrm{d}x + (x^3 e^{x^2+y} + \sin y \cos y)\,\mathrm{d}y = 0$.

   Solution: The equation is of the form

   $$a(x,y)\,\mathrm{d}x + b(x,y)\,\mathrm{d}y = 0,$$

   where

   $$a(x,y) = x^2 e^{x^2+y}(2x^2+3), \qquad \text{and} \qquad b(x,y) = x^3 e^{x^2+y} + \sin y \cos y.$$

   We have

   $$\frac{\partial a}{\partial y} = x^2 e^{x^2+y}(2x^2+3), \qquad \text{and} \qquad \frac{\partial b}{\partial x} = 3x^2 e^{x^2+y} + x^3 \cdot 2x e^{x^2+y} + 0,$$

   so the equation is *exact*. This means that there is a function $F(x,y)$ such that

   $$\frac{\partial F(x,y)}{\partial x} = a(x,y), \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = b(x,y).$$

   The latter condition gives

   $$F(x,y) = \int b(x,y)\mathrm{d}y = \int (x^3 e^{x^2+y} + \sin y \cos y)\mathrm{d}y = x^3 e^{x^2+y} + \frac{1}{2}\sin^2 y + f(x),$$

   and substituting this into the former, we get

   $$\frac{\partial F(x,y)}{\partial x} = 3x^2 e^{x^2+y} + 2x^4 e^{x^2+y} + f'(x) = a(x,y) = x^2 e^{x^2+y}(2x^2+3).$$

   From this it is clear that we can choose $f(x) = 0$, and we get

   $$F(x,y) = x^3 e^{x^2+y} + \frac{1}{2}\sin^2 y.$$

   Finally, we can write the solution in implicit form as

   $$x^3 e^{x^2+y} + \frac{1}{2}\sin^2 y = A,$$

   with A an arbitrary constant.

2. Solve the initial value problem $y' + y\sin x = y^2 \sin x$ with $y(\frac{\pi}{2}) = \frac{1}{2}$.

   Solution: We recognize the equation as a Bernoulli equation $y' + \alpha y = \beta y^k$, with $k = 2$, and recall that a good substitution is $u = y^{1-k} = y^{-1}$ or $y = \frac{1}{u}$. If one does not remember the exact exponent q in the substitution $y = u^q$, then it is also possible to work with an unknown exponent q until a point when the equation itself reveals which value of q would lead to a linear equation.

Getting back to solving the equation, if $y = \frac{1}{u}$, then $y' = -\frac{u'}{u^2}$, and so

$$-\frac{u'}{u^2} + \frac{\sin x}{u} = \frac{\sin x}{u^2}, \qquad \text{or} \qquad u' - u\sin x = -\sin x.$$

Since $(\cos x)' = -\sin x$, a suggested integrating factor is $\mu(x) = e^{\cos x}$. Let us compute

$$(e^{\cos x}u(x))' = e^{\cos x}u'(x) + e^{\cos x}(-\sin x)u(x) = e^{\cos x}(u'(x) - u(x)\sin x).$$

This must be equal to $-e^{\cos x}\sin x$, if u were to satisfy $u' - u\sin x = -\sin x$. So we have

$$(e^{\cos x}u(x))' = -e^{\cos x}\sin x.$$

A direct integration gives

$$e^{\cos x}u(x) = -\int e^{\cos x}\sin x\,\mathrm{d}x = \int e^{\cos x}\,\mathrm{d}\cos x = e^{\cos x} + C,$$

and thus

$$u(x) = 1 + Ce^{-\cos x}.$$

Before converting this into an expression for $y(x)$, we want to impose the initial condition $y(\frac{\pi}{2}) = \frac{1}{2}$. In terms of $u = \frac{1}{y}$, this initial condition is $u(\frac{\pi}{2}) = 2$. Since $\cos\frac{\pi}{2} = 0$, it leads to

$$u(\tfrac{\pi}{2}) = 1 + C = 2, \qquad \text{that is,} \qquad C = 1.$$

Finally, the solution to the original initial value problem is

$$y(x) = \frac{1}{u(x)} = \frac{1}{1 + e^{-\cos x}}.$$