Math 315 Midterm Exam Winter 2014
Version 3

1. Solve $(3y\cos x + xe^x + e^x)\mathrm{d}x + (3\sin x + 2y)\mathrm{d}y = 0$.

   Solution: The equation is of the form

$$a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y = 0,$$

where

$$a(x,y) = 3y\cos x + xe^x + e^x, \qquad \text{and} \qquad b(x,y) = 3\sin x + 2y.$$

We have

$$\frac{\partial a}{\partial y} = 3\cos x + 0, \qquad \text{and} \qquad \frac{\partial b}{\partial x} = 3\cos x + 0,$$

so the equation is *exact*. This means that there is a function $F(x,y)$ such that

$$\frac{\partial F(x,y)}{\partial x} = a(x,y), \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = b(x,y).$$

The latter condition gives

$$F(x,y) = \int b(x,y)\mathrm{d}y = \int (3\sin x + 2y)\mathrm{d}y = 3y\sin x + y^2 + f(x),$$

and substituting this into the former, we get

$$\frac{\partial F(x,y)}{\partial x} = 3y\cos x + f'(x) = a(x,y) = 3y\cos x + xe^x + e^x.$$

Now we need to find a function $f(x)$ such that $f'(x) = xe^x + e^x$. This can be done either by the direct observation $(xe^x)' = e^x + xe^x$, or by integration

$$f(x) = \int (xe^x + e^x)\,\mathrm{d}x = xe^x + C,$$

which can be computed with the help of integration by parts

$$\int xe^x\mathrm{d}x = \int x\,\mathrm{d}e^x = xe^x - \int e^x\mathrm{d}x = xe^x - e^x + C.$$

Since we just need one function $f(x)$ such that $f'(x) = xe^x + e^x$, we pick the simplest choice $f(x) = xe^x$, which yields

$$F(x,y) = 3y\sin x + y^2 + xe^x.$$

Finally, we can write the solution in implicit form as

$$3y\sin x + y^2 + xe^x = A,$$

with A an arbitrary constant.

2. Solve the initial value problem $xy' = xe^{-y/x} + y$ with $y(1) = 0$.

Solution: Let us write the equation as

$$y' = e^{-y/x} + y/x.$$

We immediately recognize this as a *homogeneous equation*, because the right hand side $f(x, y) = e^{-y/x} + y/x$ satisfies

$$f(\lambda x, \lambda y) = e^{-(\lambda y)/(\lambda x)} + \frac{\lambda y}{\lambda x} = e^{-y/x} + \frac{y}{x} = f(x, y), \qquad \text{for any } \lambda \neq 0.$$

We are going to carry out the substitution $y = ux$, where $u = u(x)$ is the new dependent unknown that replaces $y(x)$. Note that in terms of u, the initial condition $y(1) = 0$ becomes $u(1) = y(1)/1 = 0$. Taking into account $y' = u'x + u$, we get

$$u'x + u = e^{-u} + u.$$

After some straightforward manipulations, we end with the separable equation

$$e^u u' = \frac{1}{x},$$

which yields

$$e^{u(x)} = \log|x| + C.$$

At this point, it is convenient to use the initial condition $u(1) = 0$, that is,

$$\underbrace{e^0}_{=1} = \underbrace{\log 1}_{=0} + C,$$

to settle the value $C = 1$ for C. Therefore, we can write

$$u(x) = \log(\log|x| + 1),$$

and so the final solution is

$$y(x) = xu(x) = x\log(\log|x| + 1).$$