MATH 315 WINTER 2014 MIDTERM EXAM

FEEDBACK TO THE STUDENT

The overall performance of the class at the midterm exam was impressive: average 13.51 out of 15 points (with version averages 13.29 for Version 1, 14.00 for Version 2, and 13.27 for Version 3). It was great to see so many original thoughts: Problems were (correctly) solved in wildly different ways. For example, some students solved the Bernoulli equation by direct integration. However, apart from the fact that exact equations and the Bernoulli equation seemed to be new to a few students, there were some errors that occur frequently, and some errors that are not so common but quite serious. I would like to point those out here and give suggestions on how they can be rectified.

Explanations. The first point to be mentioned is that many students simply write formulas without any explanation whatsoever. Please use words to explain what you are doing. Make sure you do not leave any doubt in the grader's mind as to whether you know the particular topic or technique under discussion. For an example of how you can combine words with formulas, have a look at the midterm solutions posted on the course webpage.

Careless errors. A few students made a mistake when copying down the problem, or when copying an expression from one line to the next. One student "solved" the problem $y' = f(x)$ by differentiating $f(x)$, rather than integrating. These are errors that can easily be avoided with a bit of focus.

Elementary math. The following *false* identities were used by some students:

$$\begin{aligned}\log(a+b) &= \log a + \log b,\\ \log\log a &= (\log a)^2,\\ e^x e^x &= e^{x^2},\\ \frac{1}{a+b} &= \frac{1}{a} + \frac{1}{b}.\end{aligned}$$

Among those, the last one was the most common, which was used to find $y(x)$ from

$$\frac{1}{y(x)} = f(x) + g(x),$$

and to arrive at

$$y(x) = \frac{1}{f(x)} + \frac{1}{g(x)}.$$

If you made one of these errors, please review the basic properties of fractions, logarithms, and exponential functions.

Integration and differentiation. A few students made a sign error when integrating $\sin x$ or $\cos x$. You can easily figure out these signs if you remember how the graphs of $\sin x$ and $\cos x$ look like, or if you know how these functions behave near $x = 0$. For example, $\cos x$ is decreasing for $x > 0$ small, so $(\cos x)'$ must be negative there, and since $\sin x$ is positive for $x > 0$ small, we deduce that $(\cos x)' = -\sin x$.

One student wrote that the derivative of $\frac{1}{x}$ is $\log x$. Probably it was a careless error. If it wasn't, note that $(\log x)' = \frac{1}{x}$ and $(x^a)' = ax^{a-1}$ for all real numbers a. In particular, putting $a = -1$ into the latter formula gives the derivative of $\frac{1}{x}$.

Substitutions. In an attempt to solve a homogeneous equation, a student suggested to use the substitution $u = y/x$, and wrote

$$u = \frac{y}{x} \qquad \Rightarrow \qquad \mathrm{d}u = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

It seems that the correct expression should have been

$$u = \frac{y}{x} \qquad \Rightarrow \qquad u' = \frac{y'x - y}{x^2}, \tag{$*$}$$

or

$$u = \frac{y}{x} \qquad \Rightarrow \qquad \mathrm{d}u = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2}.$$

In any case, since the original equation $y' = f(x, y)$ is given in terms of y and y', and u is the new unknown that is supposed to replace y, a more convenient strategy would be to write the substitution as $y = ux$, and find y' as

$$y' = u'x + u.$$

This way, we can plug y and y' into the original equation directly, and obtain an equation in terms of u, u', and x. On the other hand, the formula $(*)$ would have been useful if we were replacing the unknown u by a new unknown y.

Exact equations. There were a couple of instances of "solving" the equation

$$a(x, y)\mathrm{d}x + b(x, y)\mathrm{d}y = 0, \tag{$**$}$$

by direct integration, as

$$\int a(x, y)\mathrm{d}x + \int b(x, y)\mathrm{d}y = 0. \tag{$\dagger$}$$

This resulted in wrong results such as integrating

$$\cos x \,\mathrm{d}x + 2y \sin x \,\mathrm{d}y = 0,$$

by the "rule" $(\dagger)$, and getting

$$\sin x + y^2 \sin x + C = 0.$$

Note that $(\dagger)$ would work only if the equation was separable, that is, if $a(x, y)$ depended only on x and $b(x, y)$ depended only on y.

The last item on our list is not so much an error as a flaw in presentation. If the equation $(**)$ is exact, we find a function $F(x, y)$ such that

$$\frac{\partial F(x, y)}{\partial x} = a(x, y), \qquad \text{and} \qquad \frac{\partial F(x, y)}{\partial y} = b(x, y).$$

Some students gave a formula for such $F(x, y)$, and considered that the problem was solved. The original problem is of the form "Find all solutions of ...", hence it would be appropriate to give an explicit formula for the solutions, or to describe in an implicit fashion how to find these solutions. For example, the following is an acceptable description.

> All solutions of $(**)$ are given implicitly by
>
> $$xy^3 + \sin x = A,$$
>
> where A is an arbitrary constant.