THE LAPLACE TRANSFORM

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Abstract. We introduce the Laplace transform, and use it to solve initial value problems for constant coefficient linear differential equations.

Contents

1. Introduction

We know that the logarithm function turns multiplication into addition, in the sense that

$$\log(a \cdot b) = \log a + \log b, \tag{1}$$

for any positive numbers a and b. Since addition is much easier than multiplication when performed manually, before the advent of electronic calculators, it was a common practice to multiply numbers with many digits by using the formula

$$a \cdot b = e^{\log(a\cdot b)} = e^{\log a + \log b}, \tag{2}$$

where the logarithms and exponentials are looked up in a table of logarithms.

In the same vein, some procedures on *functions* transform differentiation into a simple operation. The Laplace transform is an example of such a procedure.

Definition 1. Given a function $f(t)$ defined for $t \geq 0$, its *Laplace transform* $F(s)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t := \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t, \tag{3}$$

whenever $F(s)$ is finite. We also call the operation of transforming $f(t)$ into $F(s)$ the *Laplace transform*, and write

$$\mathscr{L}\{f(t)\} = F(s). \tag{4}$$

Remark 2. In these notes, we will mostly think of the variable s as real valued, although it makes perfect sense to consider complex values for s.

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Remark 3. Suppose that the function $f(t)$ satisfies the condition

$$|f(t)| \leq Me^{\alpha t}, \qquad \text{for all} \quad t \geq 0, \tag{5}$$

with some (real) constants M and α. Then the Laplace transform $F(s)$ given by (3) is well-defined for all values $s > \alpha$. Indeed, fixing $s > \alpha$ and letting $\beta = s - \alpha > 0$, we have

$$|e^{-st} f(t)| \leq Me^{-st} e^{\alpha t} = Me^{-\beta t}, \tag{6}$$

and so

$$\Big| \int_T^\infty e^{-st} f(t)\, \mathrm{d}t \Big| \leq \int_T^\infty |e^{-st} f(t)|\, \mathrm{d}t \leq M \int_T^\infty e^{-\beta t}\, \mathrm{d}t = \frac{M}{\beta} e^{-\beta T}, \tag{7}$$

which tends to 0 as $T \to \infty$, showing that the limit in (3) exists. In fact, the same argument implies that $F(s)$ is well-defined for all complex values of s satisfying $\operatorname{Re} s > \alpha$.

Example 4. Let us compute the Laplace transform of $f(t) = e^{at}$, where a is a real constant. By definition, we have

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, \mathrm{d}t = \int_0^\infty e^{(a-s)t}\, \mathrm{d}t. \tag{8}$$

If $s = a$, then

$$\int_0^\infty e^{(a-s)t}\, \mathrm{d}t = \int_0^\infty \mathrm{d}t = t\Big|_0^\infty = \lim_{T\to\infty} (T - 0) = \infty, \tag{9}$$

and if $s \neq a$, then

$$\int_0^\infty e^{(a-s)t}\, \mathrm{d}t = \frac{e^{(a-s)t}}{a-s}\Bigg|_0^\infty = \frac{1}{a-s}\left(\lim_{T\to\infty} e^{(a-s)T} - 1 \right) = \begin{cases} \frac{1}{s-a}, & \text{for } s > a, \\ \infty, & \text{for } s < a. \end{cases} \tag{10}$$

We conclude that

$$\mathscr{L}\{e^{at}\} = F(s) = \frac{1}{s-a}, \qquad (s > a). \tag{11}$$

In a certain sense, the number a measures how fast the function e^{at} grows or decays, say, compared to functions of the form $e^{\alpha t}$ with different constants α. This rate is "detected" by the Laplace transform, in the form of a singularity at $s = a$. In other words, supposing that we do not know the value of a in $f(t) = e^{at}$, we can recover it from the location of the singularity of $F(s) = \mathscr{L}\{f(t)\}$, see Figure 1.

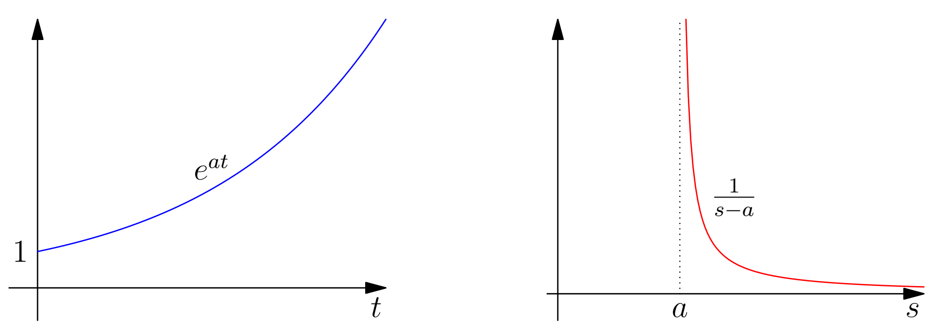

FIGURE 1. The function e^{at} and its Laplace transform $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$.

Example 5. As a slight extension of Example 4, let us compute the Laplace transform of $f(t) = e^{(a+bi)t}$, where a and b are real constants. By definition, we have

$$\mathscr{L}\{e^{(a+bi)t}\} = \int_0^\infty e^{-st} e^{(a+bi)t}\, \mathrm{d}t = \int_0^\infty e^{(a+bi-s)t}\, \mathrm{d}t. \tag{12}$$

Since the case $b = 0$ is already considered in Example 4, let us assume $b \neq 0$, which implies that $a + bi - s \neq 0$. Hence we have

$$\mathscr{L}\{e^{(a+bi)t}\} = \int_0^\infty e^{(a+bi-s)t}\,\mathrm{d}t = \left.\frac{e^{(a+bi-s)t}}{a+bi-s}\right|_0^\infty = \left.\frac{e^{(a-s)t}(\cos(bt) + i\sin(bt))}{a+bi-s}\right|_0^\infty, \tag{13}$$

where we have used the Euler formula $e^{(a+bi-s)t} = e^{(a-s)t}(\cos(bt) + i\sin(bt))$. If $s = a$, as the functions $\cos(bt)$ and $\sin(bt))$ oscillate between -1 and 1, the limit of $\cos(bt) + i\sin(bt)$ as $t \to \infty$ does not exist, meaning that $F(s) = \mathscr{L}\{e^{(a+bi)t}\}$ is *not* defined at $s = a$. Similarly, $F(s)$ is *not* defined for $s < a$, because in this case $e^{(a-s)t} \to \infty$ as $t \to \infty$. Finally, for $s > a$ we have $e^{(a-s)t} \to 0$ as $t \to \infty$, implying that

$$\mathscr{L}\{e^{(a+bi)t}\} = \frac{1}{s-a-bi}, \qquad (s > a). \tag{14}$$

Although we have assumed that $b \neq 0$ in deriving this formula, it is also true in the case $b = 0$, because we have proved it already, as (11).

Example 6. Continuing the preceding example, let us compute $F(s) = \mathscr{L}\{e^{(a+bi)t}\}$ for complex values of s. Let $s = x + yi$ with x and y real numbers. We have

$$F(x+yi) = \int_0^\infty e^{-(x+yi)t}e^{(a+bi)t}\,\mathrm{d}t = \int_0^\infty e^{(a-x+(b-y)i)t}\,\mathrm{d}t. \tag{15}$$

This is just (12) with x in place of s and $b-y$ in place of b. In other words, $F(x+yi)$ is equal to $\mathscr{L}\{e^{(a+(b-y)i)t}\}$ evaluated at $s = x$, and so

$$F(x+yi) = \frac{1}{x+yi-a-bi}, \qquad (x > a), \tag{16}$$

that is,

$$F(s) = \mathscr{L}\{e^{(a+bi)t}\} = \frac{1}{s-a-bi}, \qquad (\operatorname{Re} s > a). \tag{17}$$

We can extend the observation we made in Example 4, as follows. The Euler formula yields

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt)), \tag{18}$$

meaning that while the number a corresponds to how fast the function $e^{(a+bi)t}$ *grows or decays*, the number b measures how fast it *oscillates*. Its Laplace transform $F(s) = \frac{1}{s-a-bi}$ as a function of a complex variable s, has a singularity at $s = a + bi$. We see that the location of the singularity of $F(s)$ reveals information about growth and oscillatory behaviour of $f(t)$: Supposing that the singularity is at $s = a + bi$, the real part a corresponds to the growth rate, and the imaginary part b corresponds to the frequency of oscillation, see Figure 2.

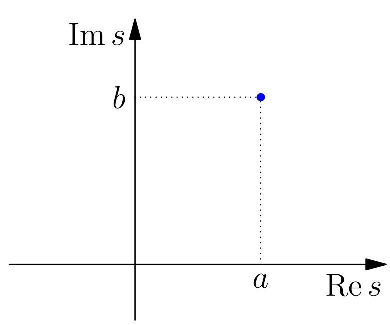

FIGURE 2. The Laplace transform of $e^{(a+bi)t}$ is singular at $s = a + bi$.

2. Linearity

One of the most important properties of the Laplace transform is *linearity*. Suppose that

$$F(s) = \mathscr{L}\{f(t)\}, \qquad \text{and} \qquad G(s) = \mathscr{L}\{g(t)\}, \tag{19}$$

are well-defined, for some value of s. Then for the same value of s, and for any real or complex constants λ and μ, the Laplace transform $\mathscr{L}\{\lambda f(t) + \mu g(t)\}$ is well-defined, and we have

$$\begin{aligned}\mathscr{L}\{\lambda f(t) + \mu g(t)\} &= \int_0^\infty e^{-st}(\lambda f(t) + \mu g(t))\,\mathrm{d}t = \int_0^\infty (\lambda e^{-st} f(t) + \mu e^{-st} g(t))\,\mathrm{d}t \\ &= \lambda \int_0^\infty e^{-st} f(t)\,\mathrm{d}t + \mu \int_0^\infty e^{-st} g(t)\,\mathrm{d}t \\ &= \lambda \mathscr{L}\{f(t)\} + \mu \mathscr{L}\{g(t)\},\end{aligned}$$

where we have used linearity of integrals (and limits) in the second line. To reiterate, we have

$$\mathscr{L}\{\lambda f(t) + \mu g(t)\} = \lambda \mathscr{L}\{f(t)\} + \mu \mathscr{L}\{g(t)\}. \tag{20}$$

Example 7. Since

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \tag{21}$$

linearity and (17) give us a simple way to compute the Laplace transform of $\cos t$.

$$\begin{aligned}\mathscr{L}\{\cos t\} &= \frac{1}{2}\mathscr{L}\{e^{it}\} + \frac{1}{2}\mathscr{L}\{e^{-it}\} = \frac{1}{2}\cdot\frac{1}{s-i} + \frac{1}{2}\cdot\frac{1}{s+i} = \frac{1}{2}\cdot\frac{s+i+s-i}{(s-i)(s+i)} \\ &= \frac{s}{s^2+1}.\end{aligned}$$

This formula merits a special display:

$$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}. \tag{22}$$

Note that $\mathscr{L}\{\cos t\}$ has two singularities located at i and $-i$, wee Figure 5.

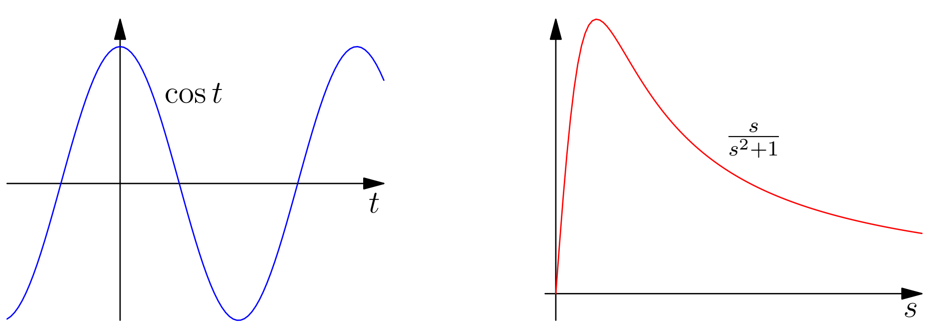

Figure 3. The function $\cos t$ and its Laplace transform $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$.

Example 8. Similarly, taking into account

$$\sin t = \frac{e^{it} - e^{-it}}{2i}, \tag{23}$$

we compute the Laplace transform of $\sin t$ as

$$\begin{aligned}\mathscr{L}\{\sin t\} &= \frac{1}{2i}\mathscr{L}\{e^{it}\} - \frac{1}{2i}\mathscr{L}\{e^{-it}\} = \frac{1}{2i}\cdot\frac{1}{s-i} - \frac{1}{2i}\cdot\frac{1}{s+i} = \frac{1}{2i}\cdot\frac{s+i-(s-i)}{(s-i)(s+i)} \\ &= \frac{1}{s^2+1}.\end{aligned}$$

To conclude, we have

$$\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}. \tag{24}$$

Note that $\mathscr{L}\{\sin t\}$ has two singularities located at i and $-i$, wee Figure 5.

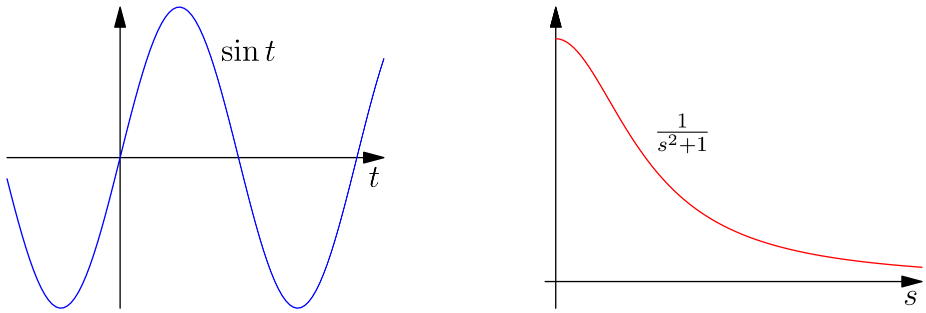

FIGURE 4. The function $\sin t$ and its Laplace transform $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$.

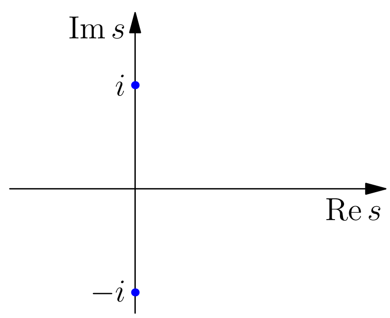

FIGURE 5. Both $\mathscr{L}\{\cos t\}$ and $\mathscr{L}\{\sin t\}$ have singularities located at i and $-i$.

3. Scaling

Another simple property of the Laplace transform is that of *scaling*. Suppose that

$$F(s) = \mathscr{L}\{f(t)\}, \tag{25}$$

and compute

$$\begin{aligned} \mathscr{L}\{f(nt)\} &= \int_0^\infty e^{-st} f(nt)\,\mathrm{d}t = \frac{1}{n}\int_0^\infty e^{-(s/n)x} f(x)\,\mathrm{d}x \\ &= \frac{1}{n}F\big(\frac{s}{n}\big), \end{aligned} \tag{26}$$

where we have introduced the new integration variable $x = nt$, and assumed that $F(\frac{s}{n})$ is well-defined. The upshot us that $\mathscr{L}\{f(nt)\}$ is well-defined for all values of s for which $F(\frac{s}{n})$ is well-defined, and is given by

$$\mathscr{L}\{f(nt)\} = \frac{1}{n}F\big(\frac{s}{n}\big). \tag{27}$$

Example 9. We can compute

$$\mathscr{L}\{\sin nt\} = \frac{1}{n} \cdot \frac{1}{(s/n)^2+1} = \frac{n}{s^2+n^2}, \tag{28}$$

and

$$\mathscr{L}\{\cos nt\} = \frac{1}{n} \cdot \frac{s/n}{(s/n)^2+1} = \frac{s}{s^2+n^2}. \tag{29}$$

Note that the singularities of $\mathscr{L}\{\cos nt\}$ and $\mathscr{L}\{\sin nt\}$ are located at $\pm ni$.

4. Differentiation in s-space

Let us differentiate

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t, \tag{30}$$

with respect to s under the integral sign, and get

$$F'(s) = \int_0^\infty (-t)e^{-st} f(t)\,\mathrm{d}t = -\mathscr{L}\{tf(t)\}. \tag{31}$$

We can interpret it as a way to differentiate $F(s)$, or as a formula to compute the Laplace transform of $tf(t)$ in terms of the Laplace transform of $f(t)$:

$$\mathscr{L}\{tf(t)\} = -F'(s). \tag{32}$$

Remark 10. The differentiation under the integral sign in (31) can be justified, if we assume that $f(t)$ satisfies the condition $|f(t)| \leq Me^{\alpha t}$, as we have in Remark 3. However, we will not go into details here.

Example 11. We can compute the Laplace transform of t as

$$\mathscr{L}\{t\} = \mathscr{L}\{t \cdot 1\} = -(\frac{1}{s})' = \frac{1}{s^2}. \tag{33}$$

More generally, applying the rule (32) repeatedly, we can compute the Laplace transform of any integer power of t, as follows.

$$\begin{aligned}\mathscr{L}\{t^n\} = \mathscr{L}\{t \cdot t^{n-1}\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{t^{n-1}\} = \ldots = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathscr{L}\{1\} \\ = (-1)^n (s^{-1})^{(n)} = (-1)^n(-1)^n n! s^{-n-1} = \frac{n!}{s^{n+1}}.\end{aligned}$$

Let us display the result on its own for later reference.

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}. \tag{34}$$

Note that the singularity of $\mathscr{L}\{t^n\}$ is located at $s = 0$ regardless of what n is, but the strength of the singularity becomes stronger with increasing n, since for instance, $\frac{1}{s^2}$ grows much faster than $\frac{1}{s}$ as $s \to 0$.

Example 12. Similarly, we have

$$\mathscr{L}\{t \sin t\} = -(\frac{1}{s^2+1})' = \frac{2s}{(s^2+1)^2}, \tag{35}$$

and

$$\mathscr{L}\{t \cos t\} = -(\frac{s}{s^2+1})' = -\frac{s^2+1-s \cdot 2s}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}. \tag{36}$$

The singularities of $\mathscr{L}\{t\cos t\}$ and $\mathscr{L}\{t\sin t\}$ are still located at $\pm i$, but the strength of the singularities became stronger, as for instance, when s is very close to i, that is, when $|s-i|$ is small, we can write

$$\mathscr{L}\{\sin t\} = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)} \approx \frac{1}{(s-i)\cdot(2i)} = \frac{1}{2i}\cdot\frac{1}{s-i}, \tag{37}$$

while

$$\mathscr{L}\{t\sin t\} = \frac{2s}{(s^2+1)^2} = \frac{2s}{(s-i)^2(s+i)^2} \approx \frac{2i}{(s-i)^2\cdot(2i)^2} = \frac{1}{2i}\cdot\frac{1}{(s-i)^2}. \tag{38}$$

5. Shift in s-space

Let s and a be some real numbers, and let us compute

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t = \int_0^\infty e^{-st} e^{at} f(t)\,\mathrm{d}t = \mathscr{L}\{e^{at} f(t)\}. \tag{39}$$

The computation is valid as long as the integral involved makes sense, that is, as long as $F(s-a)$ is well-defined. For instance, if $f(t)$ satisfies $|f(t)| \leq Me^{\alpha t}$ for some constants M and α, then (39) is valid for $s-a>\alpha$. The formula (39) can be interpreted either as a formula for $F(s-a)$, or as a way to compute the Laplace transform of $e^{at}f(t)$ in terms of the Laplace transform $F(s)$ of $f(t)$:

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a). \tag{40}$$

Example 13. Using this rule, we can easily compute

$$\mathscr{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \tag{41}$$

$$\mathscr{L}\{e^{at}\cos nt\} = \frac{s-a}{(s-a)^2+n^2}, \tag{42}$$

and

$$\mathscr{L}\{e^{at}\sin nt\} = \frac{n}{(s-a)^2+n^2}. \tag{43}$$

Note that while the singularity of $\mathscr{L}\{e^{at}t^n\}$ is at $s=a$, the singularities of $\mathscr{L}\{e^{at}\cos nt\}$ and $\mathscr{L}\{e^{at}\sin nt\}$ are located at $a \pm ni$.

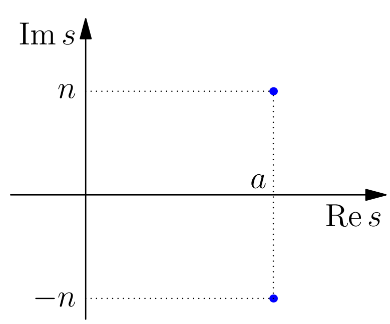

Figure 6. The singularities $\mathscr{L}\{e^{at}\cos nt\}$ and $\mathscr{L}\{e^{at}\sin nt\}$ are located at $a \pm ni$.

6. Shift in t-space

Having dealt with shift and differentiation in s-space, we now turn to shift and differentiation in t-space. For handling the shift in t-space, it will convenient to write

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_{-\infty}^\infty e^{-st}\theta(t)f(t)\,\mathrm{d}t, \tag{44}$$

where $\theta(t)$ is the *Heaviside step function*

$$\theta(t) = \begin{cases} 0, & \text{for} \quad t < 0, \\ 1, & \text{for} \quad t \geq 0. \end{cases} \tag{45}$$

Since the integral in (44) is now over the whole real line $-\infty < t < \infty$, we can freely shift the argument of the integrand, resulting in

$$\begin{aligned} \int_{-\infty}^\infty e^{-st}\theta(t)f(t)\,\mathrm{d}t &= \int_{-\infty}^\infty e^{-s(t-a)}\theta(t-a)f(t-a)\,\mathrm{d}t \\ &= e^{as}\int_{-\infty}^\infty e^{-st}\theta(t-a)f(t-a)\,\mathrm{d}t \\ &= e^{as}\int_a^\infty e^{-st}\theta(t-a)f(t-a)\,\mathrm{d}t, \end{aligned}$$

for any fixed number $a \in \mathbb{R}$, where in the third line we have used the fact that $\theta(t-a) = 0$ for $t < a$. If one wants to be pedantic, what we have done can be thought of as changing the integration variable to $r = t + a$, meaning that $t = r - a$ and $\mathrm{d}t = \mathrm{d}r$, to write

$$\int_{-\infty}^\infty e^{-st}\theta(t)f(t)\,\mathrm{d}t = \int_{-\infty}^\infty e^{-s(r-a)}\theta(r-a)f(r-a)\,\mathrm{d}r, \tag{46}$$

and then renaming r to t, as

$$\int_{-\infty}^\infty e^{-s(r-a)}\theta(r-a)f(r-a)\,\mathrm{d}r = \int_{-\infty}^\infty e^{-s(t-a)}\theta(t-a)f(t-a)\,\mathrm{d}t, \tag{47}$$

where of course, this new variable t is different than the original variable t. In any case, for $a \geq 0$, we have

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \int_{-\infty}^\infty e^{-st}\theta(t)f(t)\,\mathrm{d}t = e^{as}\int_a^\infty e^{-st}\theta(t-a)f(t-a)\,\mathrm{d}t \\ &= e^{as}\int_0^\infty e^{-st}\theta(t-a)f(t-a)\,\mathrm{d}t = e^{as}\mathscr{L}\{\theta(t-a)f(t-a)\}, \end{aligned}$$

implying that

$$\mathscr{L}\{\theta(t-a)f(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}. \tag{48}$$

Note that shift in t-space only introduces a decay factor of e^{-as} in the s-space. In particular, it does not change the locations and strengths of the singularities of the Laplace transform.

Example 14. Let us compute the Laplace transform of $\theta(t-a)$, with a constant $a \geq 0$. Invoking (48), we have

$$\mathscr{L}\{\theta(t-a)\} = \mathscr{L}\{\theta(t-a)\cdot 1\} = e^{-as}\mathscr{L}\{1\} = \frac{e^{-as}}{s}. \tag{49}$$

Note that $\mathscr{L}\{\theta(t)\} = \mathscr{L}\{1\}$, because the Laplace transform $\mathscr{L}\{f(t)\}$ cannot "see" the values of $f(t)$ for $t < 0$, and hence cannot distinguish between $\theta(t)$ and 1.

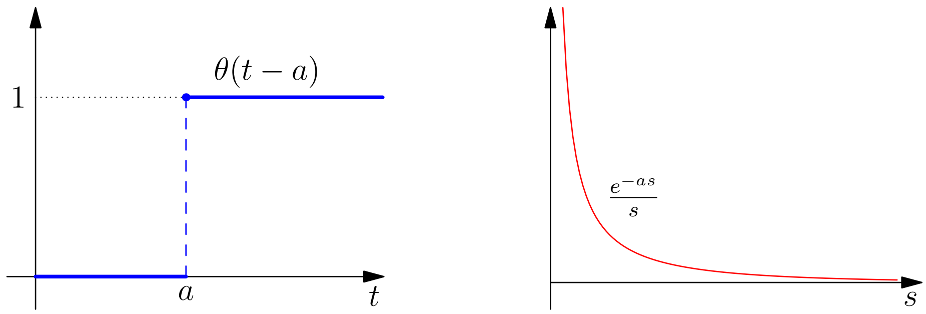

FIGURE 7. The function $\theta(t-a)$ and its Laplace transform $\mathscr{L}\{\theta(t-a)\} = \frac{e^{-as}}{s}$.

Example 15. Let a and b be real numbers satisfying $0 \le a < b$, and let

$$h(t) = \begin{cases} 0, & \text{for} \quad t < a, \\ 1, & \text{for} \quad a \le t < b, \\ 0, & \text{for} \quad t \ge b, \end{cases} \tag{50}$$

be the "rectangular pulse" between a and b. We can write

$$h(t) = \theta(t-a) - \theta(t-b), \tag{51}$$

and taking into account the preceding example, we have

$$\mathscr{L}\{h(t)\} = \mathscr{L}\{\theta(t-a)\} - \mathscr{L}\{\theta(t-b)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}. \tag{52}$$

Note that there is no singularity at $s = 0$, because $e^{-as} - e^{-bs} \approx (b-a)s$ for $s \approx 0$.

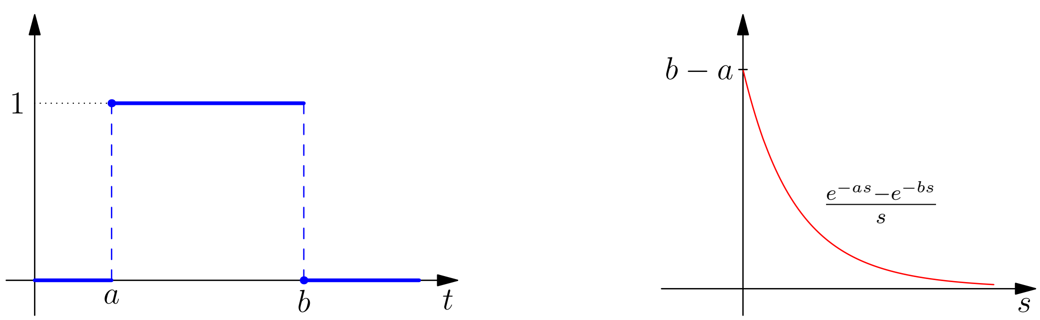

FIGURE 8. The "rectangular pulse" $\theta(t-a) - \theta(t-b)$ and its Laplace transform.

Example 16. Let us compute the Laplace transform of

$$h(t) = \begin{cases} t, & \text{for} \quad t < 3, \\ \exp(\frac{t}{5}), & \text{for} \quad 3 \le t < 2\pi, \\ \sin t, & \text{for} \quad t \ge 2\pi. \end{cases} \tag{53}$$

In terms of "rectangular pulses" and step functions, we have

$$h(t) = (1 - \theta(t-3))t + (\theta(t-3) - \theta(t-2\pi))\exp(\tfrac{t}{5}) + \theta(t-2\pi)\sin t, \tag{54}$$

Consider, for instance, the term $\theta(t-3)t$. In order to be able to apply the shift formula (48) to this term, we need to write t as $f(t-3)$ for some function f. This can be achieved by noting that $t=(t-3)+3$, so that $t=f(t-3)$ with $f(x)=x+3$. In general, supposing that $g(t)$ is a given function, we can write it as

$$g(t)=f(t-a), \qquad \text{where} \qquad f(x)=g(x+a), \tag{55}$$

and therefore

$$\mathscr{L}\{\theta(t-a)g(t)\}=\mathscr{L}\{\theta(t-a)f(t-a)\}=e^{-as}\mathscr{L}\{f(t)\}=e^{-as}\mathscr{L}\{g(t+a)\}. \tag{56}$$

By employing this formula, we compute

$$\mathscr{L}\{\theta(t-3)t\}=e^{-3s}\mathscr{L}\{t+3\}=e^{-3s}\big(\frac{1}{s^2}+\frac{1}{s}\big),$$

$$\mathscr{L}\{\theta(t-2\pi)\sin t\}=e^{-2\pi s}\mathscr{L}\{\sin(t+2\pi)\}=e^{-2\pi s}\mathscr{L}\{\sin t\}=\frac{e^{-2\pi s}}{s^2+1},$$

$$\mathscr{L}\{\theta(t-a)\exp(\tfrac{t}{5})\}=e^{-as}\mathscr{L}\{\exp(\tfrac{t+a}{5})\}=\exp(\tfrac{a}{5})\frac{e^{-as}}{s-\frac{1}{5}}=\frac{e^{-a(s-\frac{1}{5})}}{s-\frac{1}{5}}.$$

The latter result could have also been derived by using the s-space shift formula (40). Finally, piecing all this together, we have the Laplace transform of (54):

$$\begin{aligned}\mathscr{L}\{h(t)\}&=\frac{1}{s^2}-e^{-3s}\big(\frac{1}{s^2}+\frac{1}{s}\big)+\frac{e^{-3(s-\frac{1}{5})}}{s-\frac{1}{5}}-\frac{e^{-2\pi(s-\frac{1}{5})}}{s-\frac{1}{5}}+\frac{e^{-2\pi s}}{s^2+1}\\&=-\frac{e^{-3s}}{s}+\frac{1-e^{-3s}}{s^2}+\frac{e^{-3(s-\frac{1}{5})}-e^{-2\pi(s-\frac{1}{5})}}{s-\frac{1}{5}}+\frac{e^{-2\pi s}}{s^2+1}.\end{aligned}$$

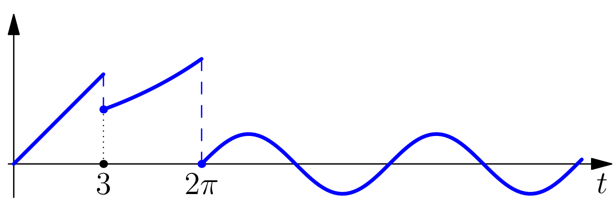

FIGURE 9. The function $h(t)$ from Example 16.

Before closing this section, we would like to display (56) again

$$\mathscr{L}\{\theta(t-a)g(t)\}=e^{-as}\mathscr{L}\{g(t+a)\}, \tag{57}$$

since it is often used when we solve differential equations.

7. Differentiation in t-space

As far as differential equations are concerned, the most important property of the Laplace transform is that it essentially turns differentiation into multiplication by s. To derive this fact, we start with the definition of $\mathscr{L}\{f'(t)$ and perform integration by parts, to get

$$\mathscr{L}\{f'(t)\}=\lim_{T\to\infty}\int_0^T e^{-st}f'(t)\,\mathrm{d}t=\lim_{T\to\infty}\Big(e^{-st}f(t)\big|_0^T+s\int_0^T e^{-st}f(t)\,\mathrm{d}t\Big). \tag{58}$$

Assume that $f(t)$ satisfies $|f(t)|\leq Me^{\alpha t}$ for all $t\geq 0$ and for some constants M and α. Then for $s>\alpha$, we have

$$|e^{-sT}f(T)|\leq Me^{-(s-\alpha)T}, \tag{59}$$

which goes to 0 as $T\to\infty$. This implies that

$$\mathscr{L}\{f'(t)\}=\lim_{T\to\infty}e^{-sT}f(T)-f(0)+s\int_0^\infty e^{-st}f(t)\,\mathrm{d}t=-f(0)+s\mathscr{L}\{f(t)\}, \tag{60}$$

for $s > \alpha$, that is

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0). \tag{61}$$

In addition to what has been assumed, if $f'(t)$ satisfies $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$, then we can apply the differentiation rule (61) to $f'(t)$, and get

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0) = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0), \tag{62}$$

for $s > \alpha$.

Example 17. We can compute

$$\mathscr{L}\{\cos t\} = \mathscr{L}\{(\sin t)'\} = s\mathscr{L}\{\sin t\} - \sin 0 = s \cdot \frac{1}{s^2+1}, \tag{63}$$

and

$$\mathscr{L}\{\sin t\} = -\mathscr{L}\{(\cos t)'\} = -s\mathscr{L}\{\cos t\} + \cos 0 = -\frac{s^2}{s^2+1} + 1 = -\frac{s^2}{s^2+1} + \frac{s^2+1}{s^2+1}, \tag{64}$$

which are consistent with what we know already.

Example 18. We know that the function $y(t) = e^{at}$ satisfies

$$y' = ay, \qquad \text{and} \qquad y(0) = 1. \tag{65}$$

Let us try to compute $\mathscr{L}\{e^{at}\}$ by using the conditions (65) only. The differential equation $y' = ay$ implies $\mathscr{L}\{y'(t)\} = a\mathscr{L}\{y(t)\}$, that is,

$$s\mathscr{L}\{y(t)\} - y(0) = a\mathscr{L}\{y(t)\}. \tag{66}$$

Taking into account $y(0) = 1$, and introducing the notation $Y(s) = \mathscr{L}\{y(t)\}$, we have

$$sY(s) - 1 = aY(s), \tag{67}$$

and so

$$Y(s) = \frac{1}{s-a}. \tag{68}$$

Example 19. In the same vein, let us compute the Laplace transform of $y(t) = \sin \omega t$ by using the conditions

$$y'' + \omega^2 y = 0, \qquad y(0) = 0, \qquad y'(0) = \omega. \tag{69}$$

So we have $\mathscr{L}\{y''(t)\} + \omega^2\mathscr{L}\{y(t)\} = 0$, and an application of (62) implies

$$s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0) + \omega^2\mathscr{L}\{y(t)\} = 0. \tag{70}$$

Taking into account the conditions $y(0) = 0$ and $y'(0) = \omega$, and introducing the notation $Y(s) = \mathscr{L}\{y(t)\}$, we have

$$s^2\, Y(s) - \omega + \omega^2 Y(s) = 0, \tag{71}$$

and so

$$Y(s) = \frac{\omega}{s^2+\omega^2}. \tag{72}$$

Exercise 20. Compute the Laplace transform of $y(t) = \cos \omega t$ by using a differential equation $\cos \omega t$ has to satisfy.

8. First order initial value problems

The Laplace transform is a convenient tool to solve initial value problems for constant coefficient linear differential equations. As we will see, it has a relative advantage especially if the right hand side of the equation is a discontinuous or a piecewise defined function.

We start with first order problems. Consider the initial value problem

$$y' + py = f(t), \qquad y(0) = \alpha, \tag{73}$$

where p and α are given numbers, and $f(t)$ is a given function. Laplace transforming the both sides of the equation, and using the differentiation rule (61), we get

$$s\mathscr{L}\{y(t)\} - y(0) + p\mathscr{L}\{y(t)\} = \mathscr{L}\{f(t)\}. \tag{74}$$

Taking into account the initial condition $y(0) = \alpha$, and introducing $Y(s) = \mathscr{L}\{y(t)\}$ and $F(s) = \mathscr{L}\{f(t)\}$, we infer

$$sY(s) - \alpha + pY(s) = F(s), \tag{75}$$

or in other words, the Laplace transform of the solution to (73) is given by

$$Y(s) = \frac{\alpha}{s+p} + \frac{F(s)}{s+p}. \tag{76}$$

It is remarkable that solving the differential equation (73) in the s-space basically amounts to dividing by $s + p$. What remains is to find the function $y(t)$ given its Laplace transform $Y(s)$. We call this operation the *inverse Laplace transform*, and write

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}. \tag{77}$$

To summarize, solving the initial value problem (73) consists of the following steps.

- Compute the Laplace transform $F(s)$ of the right hand side $f(t)$.
- Write $Y(s)$ in terms of $F(s)$, as in (76).
- Compute the inverse Laplace transform of $Y(s)$, to get the solution $y(t)$.

Example 21. Let us solve the initial value problem

$$y' + 3y = \sin t, \qquad y(0) = 2. \tag{78}$$

Laplace transforming the both sides of the equation, and using (61), we infer

$$s\mathscr{L}\{y(t)\} - y(0) + 3\mathscr{L}\{y(t)\} = \mathscr{L}\{\sin t\} = \frac{1}{s^2+1}, \tag{79}$$

which we can write as

$$(s+3)Y(s) - 2 = \frac{1}{s^2+1}, \tag{80}$$

where we have taken into account the initial condition $y(0) = 2$ and have introduced the notation $Y(s) = \mathscr{L}\{y(t)\}$. From here it is easy to find

$$Y(s) = \frac{2}{s+3} + \frac{1}{(s^2+1)(s+3)}. \tag{81}$$

Now we need to compute $y(t) = \mathscr{L}^{-1}\{Y(s)\}$. While we can easily recognize that the first term $\frac{2}{s+3}$ is the Laplace transform of $2e^{-3t}$, the second term needs some preprocessing. To this end, we use the technique of *partial fraction decompositions*, and write

$$\frac{1}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}. \tag{82}$$

We see the Laplace transforms of $\cos t$, of $\sin t$, and of e^{-3t} in the right hand side, so once we have identified the constants A, B, and C, we would be able to complete the solution. To identify these constants, we simply give a common denominator, and infer

$$\frac{1}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+3} = \frac{(As+B)(s+3)+C(s^2+1)}{(s^2+1)(s+3)}, \tag{83}$$

implying that

$$(A+C)s^2 + (3A+B)s + 3B + C = 1. \tag{84}$$

Thus we have

$$A + C = 0, \qquad 3A + B = 0, \qquad 3B + C = 1. \tag{85}$$

Let us find C and B from the first two equations, and substitute them into the third, to get

$$C = -A, \qquad B = -3A, \qquad 3\cdot(-3A) + (-A) = 1, \tag{86}$$

leading to

$$A = -\frac{1}{10}, \qquad B = -3A = \frac{3}{10}, \qquad C = -A = \frac{1}{10}. \tag{87}$$

Finally, we have

$$Y(s) = \frac{2}{s+3} - \frac{1}{10}\cdot\frac{s}{s^2+1} + \frac{3}{10}\cdot\frac{1}{s^2+1} + \frac{1}{10}\cdot\frac{1}{s+3}, \tag{88}$$

and so

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 2e^{-3t} - \frac{\cos t}{10} + \frac{3\sin t}{10} + \frac{e^{-3t}}{10} = \frac{21e^{-3t} + 3\sin t - \cos t}{10}. \tag{89}$$

Note that the method of partial fractions is actually a parallel of the method of undetermined coefficients, because finding the constants A, B, and C in (83) amounts to deciding the constants in front of e^{-3t}, $\sin t$, and $\cos t$ in the eventual solution (89). However, if the right hand side $f(t)$ was, say, a discontinuous function, then the Laplace transform method would still work, but the method of undetermined coefficients would become very complicated. In this sense, the Laplace transform method can be thought of as an extension of (or an improvement upon) the method of undetermined coefficients.

9. Second order initial value problems

Consider the initial value problem

$$y'' + py' + qy = f(t), \qquad y(0) = \alpha, \qquad y'(0) = \beta, \tag{90}$$

where p, q, α and β are given numbers, and $f(t)$ is a given function. Laplace transforming the both sides of the equation, and using the differentiation rule (62), we get

$$s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0) + ps\mathscr{L}\{y(t)\} - py(0) + q\mathscr{L}\{y(t)\} = \mathscr{L}\{f(t)\}. \tag{91}$$

Taking into account the initial conditions, and writing $Y(s) = \mathscr{L}\{y(t)\}$ and $F(s) = \mathscr{L}\{f(t)\}$, we infer

$$s^2Y(s) - s\alpha - \beta + psY(s) - p\alpha + qY(s) = F(s), \tag{92}$$

or in other words, the Laplace transform of the solution to (73) is given by

$$Y(s) = \frac{\alpha(s+p)}{s^2+ps+q} + \frac{\beta}{s^2+ps+q} + \frac{F(s)}{s^2+ps+q}. \tag{93}$$

We can now formulate a method to solve the initial value problem (90), as follows.

- Compute the Laplace transform $F(s)$ of the right hand side $f(t)$.
- Write $Y(s)$ in terms of $F(s)$, as in (93).
- Compute the inverse Laplace transform of $Y(s)$, to get the solution $y(t)$.

Example 22. Let us solve the initial value problem

$$y'' + y = \theta(t-\pi), \qquad y(0) = 0, \qquad y'(0) = 1. \tag{94}$$

Laplace transforming the both sides of the equation, and using (62), we get

$$s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0) + \mathscr{L}\{y(t)\} = \mathscr{L}\{\theta(t-\pi)\} = \frac{e^{-\pi s}}{s}. \tag{95}$$

In light of the initial conditions $y(0) = 0$ and $y'(0) = 1$, we infer

$$(s^2+1)Y(s) - 1 = \frac{e^{-\pi s}}{s}, \tag{96}$$

or equivalently

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)}, \tag{97}$$

where $Y(s) = \mathscr{L}\{y(t)\}$.

Now we need to compute $y(t) = \mathscr{L}^{-1}\{Y(s)\}$. The first term in (97) is the Laplace transform of $\sin t$. The second term is not immediately obvious, so we need to use a partial fraction decomposition. The condition

$$\frac{1}{s(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s} = \frac{(As+B)s + C(s^2+1)}{s(s^2+1)}, \tag{98}$$

leads to

$$A + C = 0, \qquad B = 0, \qquad C = 1. \tag{99}$$

Therefore we have

$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s} - e^{-\pi s}\frac{s}{s^2+1}, \tag{100}$$

and hence

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \sin t + \theta(t-\pi) - \theta(t-\pi)\cos(t-\pi) = \sin t + \theta(t-\pi)(1+\cos t), \tag{101}$$

where we have used the shift rule (48). An alternative way to write $y(t)$ is

$$y(t) = \begin{cases} \sin t, & \text{for} \quad t < \pi, \\ 1 + \sin t + \cos t, & \text{for} \quad t \geq \pi. \end{cases} \tag{102}$$

10. The inverse Laplace transform

In this section, we want to systematically derive some important properties of the inverse Laplace transform. Recall that if

$$F(s) = \mathscr{L}\{f(t)\}, \tag{103}$$

then the *inverse Laplace transform* of $F(s)$ is defined by

$$\mathscr{L}^{-1}\{F(s)\} = f(t). \tag{104}$$

In these notes, we are going to simply assume that the inverse Laplace transform is well-defined for all functions $F(s)$ that are under consideration. A justification of this fact is beyond the scope of this course.

First, let us compute the inverse Laplace transforms of a few simple functions. The fact $\mathscr{L}\{e^{(a+bi)t}\} = \frac{1}{s-a-bi}$ implies that

$$\mathscr{L}^{-1}\{\frac{1}{s-a-bi}\} = e^{(a+bi)t}. \tag{105}$$

From $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$, with the help of the substitution $m = n+1$, we can derive that

$$\mathscr{L}^{-1}\{\frac{1}{s^m}\} = \frac{t^{m-1}}{(m-1)!}. \tag{106}$$

We also have

$$\mathscr{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t, \qquad \mathscr{L}^{-1}\{\frac{s}{s^2+1}\} = \cos t, \qquad \text{and} \qquad \mathscr{L}^{-1}\{\frac{e^{-as}}{s}\} = \theta(t-a), \quad (107)$$

which are immediate from $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$, $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$, and $\mathscr{L}\{\theta(t-a)\} = \frac{e^{-as}}{s}$.

We are not going to continue deriving such formulas, since for example, the formula $\mathscr{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t$ is simply a restatement of $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$, and there is really nothing new to it. Somewhat similarly to how these formulas are derived, each operational rule for the Laplace transform such as linearity and scaling, gives rise to a rule for the inverse Laplace transform. Although these rules are again restatements of the rules for the Laplace transform, it is worth explicitly writing them down.

Linearity. We start with linearity. Let $F(s) = \mathscr{L}\{f(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$. Then by linearity of the Laplace transform, we have

$$\mathscr{L}\{\lambda f(t) + \mu g(t)\} = \lambda \mathscr{L}\{f(t)\} + \mu \mathscr{L}\{g(t)\} = \lambda F(s) + \mu G(s). \quad (108)$$

Applying the inverse Laplace transform to both sides yields

$$\lambda f(t) + \mu g(t) = \mathscr{L}^{-1}\{\lambda F(s) + \mu G(s)\}, \quad (109)$$

and plugging in $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and $g(t) = \mathscr{L}^{-1}\{G(s)\}$, we conclude that

$$\mathscr{L}^{-1}\{\lambda F(s) + \mu G(s)\} = \lambda \mathscr{L}^{-1}\{F(s)\} + \mu \mathscr{L}^{-1}\{G(s)\}, \quad (110)$$

which is the linearity property of the inverse Laplace transform.

Example 23. By linearity, we have

$$\mathscr{L}^{-1}\{\frac{5}{s^3} + \frac{2s}{s^2+1}\} = 5\mathscr{L}^{-1}\{\frac{1}{s^3}\} + 2\mathscr{L}^{-1}\{\frac{s}{s^2+1}\} = \frac{5t^2}{2} + 2\cos t, \quad (111)$$

where we have also used (106) and the second formula in (107).

Scaling. Let $F(s) = \mathscr{L}\{f(t)\}$. Then we have the scaling law

$$\mathscr{L}\{f(\lambda t)\} = \frac{1}{\lambda} F\big(\frac{s}{\lambda}\big). \quad (112)$$

Applying the inverse Laplace transform on both sides, we get

$$f(\lambda t) = \frac{1}{\lambda}\mathscr{L}^{-1}\big\{F\big(\frac{s}{\lambda}\big)\big\}. \quad (113)$$

and upon introducing $\mu = \frac{1}{\lambda}$, this is

$$\mathscr{L}^{-1}\{F(\mu s)\} = \frac{1}{\mu} f\big(\frac{t}{\mu}\big). \quad (114)$$

Remarkably, this scaling law looks formally identical to (112).

Example 24. By using the scaling law and the first formula in (107), we infer

$$\mathscr{L}^{-1}\{\frac{1}{4s^2+1}\} = \mathscr{L}^{-1}\{\frac{1}{(2s)^2+1}\} = \frac{1}{2}\sin\frac{t}{2}. \quad (115)$$

Differentiation in s-space. By applying the inverse Laplace transform to the both sides of

$$\mathscr{L}\{tf(t)\} = -F'(s), \tag{116}$$

we get

$$tf(t) = -\mathscr{L}^{-1}\{F'(s)\}. \tag{117}$$

Then plugging in $f(t) = \mathscr{L}^{-1}\{F(s)\}$, we derive the differentiation rule

$$\mathscr{L}^{-1}\{F'(s)\} = -t\mathscr{L}^{-1}\{F(s)\}. \tag{118}$$

Example 25. We have

$$\left(\frac{1}{s^2+1}\right)' = -\frac{2s}{(s^2+1)^2}, \qquad \text{that is,} \qquad \frac{s}{(s^2+1)^2} = -\frac{1}{2}\left(\frac{1}{s^2+1}\right)'. \tag{119}$$

In light of (118), this gives

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = -\frac{1}{2}\mathscr{L}^{-1}\left\{\left(\frac{1}{s^2+1}\right)'\right\} = \frac{t}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{t\sin t}{2}. \tag{120}$$

Example 26. We have

$$\left(\frac{s}{s^2+1}\right)' = \frac{1-s^2}{(s^2+1)^2}, \qquad \text{that is,} \qquad \frac{2}{(s^2+1)^2} = \left(\frac{s}{s^2+1}\right)' + \frac{1}{s^2+1}. \tag{121}$$

In light of (118), this gives

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\left(\frac{s}{s^2+1}\right)'\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = -\frac{t\cos t}{2} + \frac{\sin t}{2}. \tag{122}$$

Shift in s-space. By applying the inverse Laplace transform to the both sides of

$$F(s-a) = \mathscr{L}\{e^{at}f(t)\}, \tag{123}$$

we derive the shift rule

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}. \tag{124}$$

Example 27. Sample applications of the shift rule (124) give

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^m}\right\} = \frac{e^{at}t^{m-1}}{(m-1)!}, \tag{125}$$

and

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2+1}\right\} = e^{at}\sin t. \tag{126}$$

Shift in t-space. It is straightforward to derive from

$$\mathscr{L}\{\theta(t-a)f(t-a)\} = e^{-as}\mathscr{L}\{f(t)\}, \tag{127}$$

the shift rule

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = \theta(t-a)f(t-a). \tag{128}$$

Example 28. We have

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s-k}\right\} = \theta(t-a)e^{k(t-a)}, \tag{129}$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-ks}}{(s-a)^m}\right\} = \frac{\theta(t-k)e^{a(t-k)}(t-k)^{m-1}}{(m-1)!}, \tag{130}$$

and

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}s}{s^2+1}\right\} = \theta(t-a)\cos(t-a). \tag{131}$$