FROBENIUS SERIES SOLUTIONS

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Abstract. We introduce the Frobenius series method to solve second order linear equations, and illustrate it by concrete examples.

Contents

1. Regular singular points

Consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{1}$$

where a, b, and c are polynomials, or equivalently,

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where

$$p(x) = \frac{b(x)}{a(x)}, \qquad \text{and} \qquad q(x) = \frac{c(x)}{a(x)}. \tag{3}$$

Recall that a point $x = \alpha$ is called a *singular point* of (2) if

$$|p(x)| \to \infty, \qquad \text{or} \qquad |q(x)| \to \infty, \qquad \text{as} \quad x \to \alpha. \tag{4}$$

Recall also that an *ordinary point* of (2) is a point at which the functions $p(x)$ and $q(x)$ are continuous. We have seen that one can solve the equation in terms of a power series centred at an ordinary point. In these notes, we will generalize the power series method so that we can solve the equation (2) at least near *some* singular points. The method is called the *Frobenius method*, named after the mathematician Ferdinand Georg Frobenius.

Example 1. In fact, we have already encountered an equation with a singular point, and we have solved it near its singular point. The Cauchy-Euler equation

$$x^2y'' + Pxy' + Qy = 0, \tag{5}$$

has a singular point at $x = 0$, and we know that a solution for $x > 0$ is given by

$$y(x) = x^r = e^{r \log x}, \tag{6}$$

where r is a root of the characteristic (or auxiliary) equation

$$r^2 + (P - 1)r + Q = 0. \tag{7}$$

This example shows that at least some singular points can be "tamed".

Comparing (1) with a Cauchy-Euler equation leads us to the following definition.

Date: April 4, 2014.

Definition 2. A singular point $x = \alpha$ is called a *regular singular point* of the equation (1) if (1) can be written as

$$(x-\alpha)^2\tilde{a}(x)y'' + (x-\alpha)\tilde{b}(x)y' + \tilde{c}(x)y = 0, \tag{8}$$

where $\tilde{a}(x)$, $\tilde{b}(x)$, and $\tilde{c}(x)$ are polynomials, with $\tilde{a}(\alpha) \neq 0$.

A singular point that is not a regular singular point is called a irregular singular point.

Example 3. Consider the equation

$$(5-x^2)y'' + \frac{1+x}{x}y' - \big(\frac{3}{x^2} + x\big)y = 0. \tag{9}$$

It is easy to see that the singular points are at $x = 0$ and $x = \pm 5$. Let us try to show that the point $x = 0$ is a regular singular point. We need Definition 2 with $\alpha = 0$. According to the definition, all coefficients of the equation must be polynomials. We multiply (9) by x^2, to get

$$x^2(5-x^2)y'' + x(1+x)y' - (3+x^3)y = 0, \tag{10}$$

which is of the form (8), with

$$\tilde{a}(x) = 5 - x^2, \qquad \tilde{b}(x) = 1 + x, \qquad \tilde{c}(x) = -(3+x^3). \tag{11}$$

The only remaining condition to check is the condition $\tilde{a}(\alpha) \neq 0$. Since $\tilde{a}(0) = 5 \neq 0$, this condition is true, and hence $x = 0$ is a regular singular point of (9).

Remark 4. An alternative, perhaps more direct way to identify the coefficients $\tilde{a}(x)$, $\tilde{b}(x)$, and $\tilde{c}(x)$ in (9) would be to write it as

$$\tilde{a}(x)y'' + \frac{\tilde{b}(x)}{x}y' + \frac{\tilde{c}(x)}{x^2}y = 0, \tag{12}$$

that is,

$$(5-x^2)y'' + \frac{1+x}{x}y' - \frac{3+x^3}{x^2}y = 0. \tag{13}$$

Intuitively, the point $x = 0$ is a regular singular point if the coefficient of y'' is nonzero at $x = 0$, and as $x \to 0$, the coefficients of y' and y do grow faster than x^{-1} and x^{-2}, respectively.

2. Formulation of the method

Recall for convenience that a singular point $x = \alpha$ is called a *regular singular point* of (1) if (1) can be written as

$$(x-\alpha)^2\tilde{a}(x)y'' + (x-\alpha)\tilde{b}(x)y' + \tilde{c}(x)y = 0, \tag{14}$$

where $\tilde{a}(x)$, $\tilde{b}(x)$, and $\tilde{c}(x)$ are polynomials, with $\tilde{a}(\alpha) \neq 0$.

Now, the central idea of the method we are about to see is the expectation that for $x \approx \alpha$, any solution $y(x)$ of (14) must approximately solve the Cauchy-Euler equation

$$(x-\alpha)^2\tilde{a}(\alpha)y'' + (x-\alpha)\tilde{b}(\alpha)y' + \tilde{c}(\alpha)y = 0. \tag{15}$$

Note carefully that the coefficients $\tilde{a}(x)$, $\tilde{b}(x)$, and $\tilde{c}(x)$ are evaluated at $x = \alpha$, so that $\tilde{a}(\alpha)$, $\tilde{b}(\alpha)$, and $\tilde{c}(\alpha)$ are *numbers*, not functions. Comparing (15) with (5), we identify the coefficients

$$P = \frac{\tilde{b}(\alpha)}{\tilde{a}(\alpha)}, \qquad Q = \frac{\tilde{c}(\alpha)}{\tilde{a}(\alpha)}, \tag{16}$$

and so a solution of (15) for $x > \alpha$ is given by

$$\tilde{y}(x) = (x-\alpha)^r, \tag{17}$$

where r is a root of

$$r^2 + \left(\frac{\tilde{b}(\alpha)}{\tilde{a}(\alpha)} - 1\right) r + \frac{\tilde{c}(\alpha)}{\tilde{a}(\alpha)} = 0, \tag{18}$$

which is called the *indicial equation* for (14). This function $\tilde{y}(x)$ will not in general be a solution to (14), but we expect that $\tilde{y}(x)$ will be close to being a solution. Accordingly, we look for a solution to (14) in the form

$$y(x) = (x-\alpha)^r \sum_{k=0}^{\infty} a_k (x-\alpha)^k. \tag{19}$$

Because r can be fractional or a negative number, (19) is in general not a power series. It is called a *Frobenius series.*

Finally, we can formulate the method of Frobenius series as follows.

(i) Given the equation (14) with a regular singular point at $x = \alpha$, solve the indicial equation (18) and find possible values for r. Note that if we required the normalization $\tilde{a}(\alpha) = 1$ from the beginning, the indicial equation would have been

$$r^2 + \left(\tilde{b}(\alpha) - 1\right) r + \tilde{c}(\alpha) = 0, \tag{20}$$

which is a bit simpler. Note also that another way to write the indicial equation is

$$r(r-1) + \frac{\tilde{b}(\alpha)}{\tilde{a}(\alpha)} r + \frac{\tilde{c}(\alpha)}{\tilde{a}(\alpha)} = 0. \tag{21}$$

(ii) For each possible value of r, substitute the Frobenius series (19) into (14), and find the coefficients $a_1, a_2, a_3, \ldots$ in terms of the leading coefficient a_0.

We have a theorem stating that this method works, which we recall here without proof.

Theorem 5. *The method of Frobenius series yields at least one solution to* (14)*.*

3. Examples

Example 6. Consider the equation

$$5x^2y'' + x(1+x)y' - y = 0. \tag{22}$$

Comparing it with (14), we conclude that $x = 0$ is a regular singular point, and

$$\tilde{a}(x) = 5, \qquad \tilde{b}(x) = 1 + x, \qquad \tilde{c}(x) = -1. \tag{23}$$

Keeping in mind that $\alpha = 0$, we have $\tilde{a}(\alpha) = 5$, $\tilde{b}(\alpha) = 1$, and $\tilde{c}(\alpha) = -1$. Hence the indicial equation is

$$r^2 + (\frac{1}{5} - 1)r - \frac{1}{5} = 0, \tag{24}$$

and its solutions are

$$r_{1,2} = \frac{2}{5} \pm \sqrt{\frac{4}{25} + \frac{1}{5}} = \frac{2}{5} \pm \frac{3}{5}. \tag{25}$$

Having found the possible values of r, we now proceed with substituting the Frobenius series (19) into the equation (22). Let r be either $r_1 = 1$ or $r_2 = -\frac{1}{5}$. First, write (19) as

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}, \tag{26}$$

and compute its derivatives

$$y'(x) = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r-1}, \qquad \text{and} \qquad y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r-2}. \tag{27}$$

Substituting these expressions into (22), we get

$$\sum_{k=0}^{\infty} 5(k+r)(k+r-1)a_k x^{k+r} + \sum_{k=0}^{\infty}(k+r)a_k(x^{k+r} + x^{k+r+1}) - \sum_{k=0}^{\infty} a_k x^{k+r} = 0, \tag{28}$$

which, upon some rearranging, becomes

$$\sum_{k=0}^{\infty}[5(k+r)(k+r-1) + (k+r) - 1]a_k x^{k+r} + \sum_{k=0}^{\infty}(k+r)a_k x^{k+r+1} = 0. \tag{29}$$

The coefficient of each power of x appearing in the left hand side must vanish. We observe that the first term ($k = 0$) of the first sum is a term with x^r, while the first term ($k = 0$) of the second sum is a term with x^{r+1}. In other words, the coefficient of x^r in the entire left hand side of (29) is

$$[5(0+r)(0+r-1) + (0+r) - 1]a_0 = [5r(r-1) + r - 1]a_0, \tag{30}$$

which can be read off from the $k = 0$ term of the first sum in (29). Now the twist is that $5r(r-1) + r - 1 = 0$, because r satisfies the indicial equation (24), and $5r(r-1) + r - 1 = 0$ is just (24) in disguise. This basically means that the constant a_0 can have arbitrary value, although we still have to check if there would be any constraint on a_0 induced by the conditions on the coefficients of the other powers of x in the left hand side of (29).

To find those conditions, we write the second sum in (29) as

$$\sum_{k=0}^{\infty}(k+r)a_k x^{k+r+1} = \sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r}, \tag{31}$$

where we have introduced the new variable $n = k+1$. The first sum in (29) can be written as

$$\sum_{k=0}^{\infty}[5(k+r)(k+r-1)+(k+r)-1]a_k x^{k+r} = \sum_{k=1}^{\infty}[5(k+r)(k+r-1)+(k+r)-1]a_k x^{k+r}, \tag{32}$$

since the term $k = 0$ vanishes because of the indicial equation. So the left hand side of (29) is

$$\begin{aligned}
&\sum_{k=0}^{\infty}[5(k+r)(k+r-1) + (k+r) - 1]a_k x^{k+r} + \sum_{k=0}^{\infty}(k+r)a_k x^{k+r+1} \\
&\quad = \sum_{k=1}^{\infty}[5(k+r)(k+r-1) + (k+r) - 1]a_k x^{k+r} + \sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r} \\
&\quad = \sum_{n=1}^{\infty}[5(n+r)(n+r-1) + (n+r) - 1]a_n x^{n+r} + \sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r} \\
&\qquad = \sum_{n=1}^{\infty}\big([5(n+r)(n+r-1) + (n+r) - 1]a_n + (n-1+r)a_{n-1}\big)x^{n+r},
\end{aligned}$$

where in the second step we have renamed the summation variable in the first sum in order to conveniently combine the terms of the two sums. By (29) the coefficient of each x^{n+r} must vanish, meaning that

$$[5(n+r)(n+r-1) + n + r - 1]a_n + (n-1+r)a_{n-1} = 0, \qquad n = 1, 2, \ldots, \tag{33}$$

which are the conditions we have been looking for. We can simplify it a bit, by dividing it by the common factor $n - 1 + r$, giving

$$[5(n+r) + 1]a_n + a_{n-1} = 0, \qquad n = 1, 2, \ldots, \tag{34}$$

or

$$a_n = -\frac{a_{n-1}}{5n+5r+1}, \qquad n = 1, 2, \ldots. \tag{35}$$

Finally, we can use the concrete values $r = 1$ and $r = -\frac{1}{5}$. For the case $r = 1$, we have

$$a_n = -\frac{a_{n-1}}{5n+6} = (-1)^n a_0 \prod_{k=1}^{n} (5j+1)^{-1}, \qquad n = 1, 2, \ldots, \tag{36}$$

and for $r = \frac{1}{5}$, we have

$$a_n = -\frac{a_{n-1}}{5n} = \frac{(-1)^n}{5^n n!} a_0, \qquad n = 1, 2, \ldots. \tag{37}$$

In the latter case, the solution $y(x)$ has a closed form expression

$$y(x) = x^{-\frac{1}{5}} \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n n!} a_0 x^n = a_0 x^{-\frac{1}{5}} \sum_{n=0}^{\infty} \frac{1}{n!} \big(-\frac{x}{5} \big)^n = a_0 x^{-\frac{1}{5}} e^{-\frac{x}{5}}. \tag{38}$$

Let us check if this expression indeed satisfies the equation (19). For simplicity, putting $a_0 = 1$, we compute

$$\begin{aligned} y'(x) &= -\frac{1}{5} x^{-\frac{6}{5}} e^{-\frac{x}{5}} - \frac{1}{5} x^{-\frac{1}{5}} e^{-\frac{x}{5}}, \\ y''(x) &= \frac{6}{25} x^{-\frac{11}{5}} e^{-\frac{x}{5}} + \frac{2}{25} x^{-\frac{6}{5}} e^{-\frac{x}{5}} + \frac{1}{25} x^{-\frac{1}{5}} e^{-\frac{x}{5}}. \end{aligned}$$

Now writing all the relevant terms side by side

$$\begin{aligned} 5x^2 y''(x) &= \frac{6}{5} x^{-\frac{1}{5}} e^{-\frac{x}{5}} + \frac{2}{5} x^{\frac{4}{5}} e^{-\frac{x}{5}} + \frac{1}{5} x^{\frac{9}{5}} e^{-\frac{x}{5}}, \\ x y'(x) &= -\frac{1}{5} x^{-\frac{1}{5}} e^{-\frac{x}{5}} - \frac{1}{5} x^{\frac{4}{5}} e^{-\frac{x}{5}}, \\ x^2 y'(x) &= -\frac{1}{5} x^{\frac{4}{5}} e^{-\frac{x}{5}} - \frac{1}{5} x^{\frac{9}{5}} e^{-\frac{x}{5}}, \\ -y(x) &= -x^{-\frac{1}{5}} e^{-\frac{x}{5}}, \end{aligned}$$

makes it clear that $y(x) = x^{-\frac{1}{5}} e^{-\frac{x}{5}}$ is a solution to (19).

Exercise 7. Find a solution of

$$3x^2 y'' + (2x^2 - x) y' + y = 0, \tag{39}$$

in terms of the Frobenius series

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k. \tag{40}$$